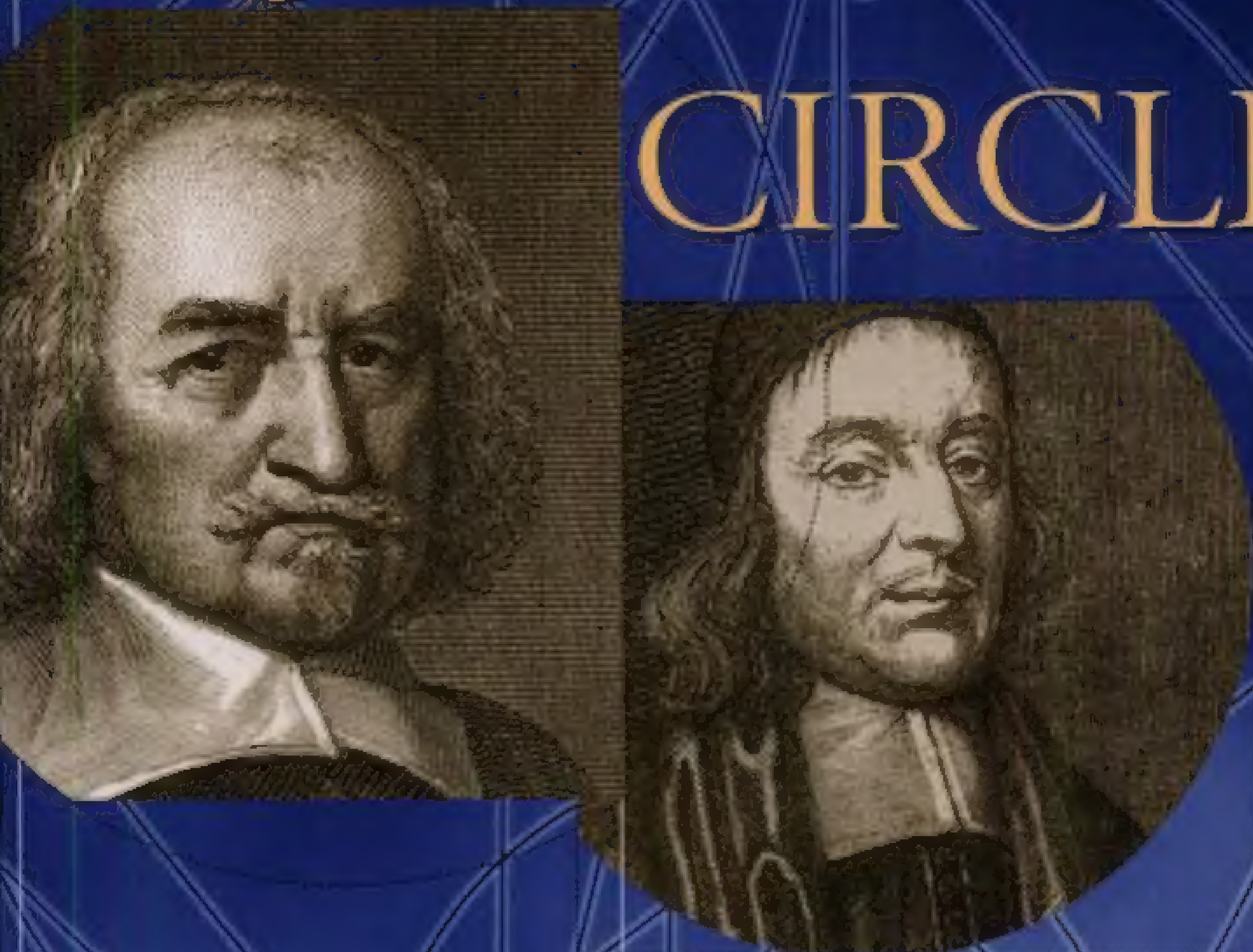
DOUGLAS M. JESSEPH

SQUARING the CIRCLE



The WAR between HOBBES and WALLIS

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Contents

Preface

This book began as a footnote. In an earlier study of Berkeley's philosophy of mathematics I was led to consider Hobbes's views on the relationship between geometry and algebra, which at the time I deemed worthy of a passing reference. My curiosity had been aroused, however, and from there I soon found myself engaged in a study of Hobbes's mathematical writings, which inevitably turned into a study of his controversy with Wallis. I certainly did not start out planning to devote a significant part of my life to the detailed examination of a controversy that began with technical issues in mathematics and grew to include questions of theology, philology, politics, and the very nature of reason. In this, I suppose, my writing of the book mirrors the controversy itself.

While conducting the research for this work I have relied upon the assistance of a number of scholars working on seventeenth-century philosophy and mathematics, and it is a pleasure to acknowledge my debts. Margaret Wilson, whose untimely death came as the project was being completed, was a frequent source of information and support. Her absence will be keenly felt by scholars of seventeenth-century philosophy. Dan Garber and Ed Curley were instrumental in a number of ways: through their writings, their conversations with me, and not least in persuading granting agencies that the world really does need to know more about Hobbes's circle-squaring efforts. Roger Ariew, Moti Feingold, and Marjorie Grene gave helpful feedback at various stages in the development of the work. Kris Heitman deserves special thanks for having read and commented on an earlier draft, and for helping me clarify my thoughts on a number of issues. Cees Leijenhorst, Paolo Mancosu, and Siegmund Probst also deserve thanks for helpful discussions and the concrete assistance they have provided. My colleagues at North Carolina State University, particularly Randy Carter, John Carroll, Joe Levine, and Tim Hinton, all read and commented on various pieces of the book, and I have always found their advice valuable. Hal Levin's aid as technology guru and keeper of the sacred mysteries

of the Mac operating system was crucial in getting the text, equations, and figures into a presentable form. I also received major assistance from Michael Mahoney, who subjected the whole typescript to a detailed and careful criticism as referee for the University of Chicago Press. Whatever flaws remain, the work has been vastly improved by his valuable suggestions. Susan Abrams, executive editor at the University of Chicago Press, was a source of encouragement throughout the project. I hope her patience has been adequately rewarded.

An undertaking like this would be impossible without support from a variety of institutions. The National Endowment for the Humanities assisted twice, first with a 1991 Summer Stipend (#FT-35703) and then with a yearlong Fellowship (#FB-3343-96) for the academic year 1996–97. The National Humanities Center provided an absolutely ideal work environment during a fellowship for the academic year 1993–94, which was generously supported by Andrew W. Mellon Foundation. The largesse of the North Carolina State University College of Humanities and Social Sciences has been considerable, amounting to no less than two summer stipends, a travel grant to support research in Paris, and research funds to enable the purchase of books and equipment. It goes without saying that my efforts have been enabled by the support and cooperation from many libraries and archives. His Grace the Duke of Devonshire and the Trustees of the Chatsworth Settlement were kind enough to grant me access to the Hobbes archives at Chatsworth House, and I must especially thank P. J. Day, the keeper of collections at Chatsworth, for his assistance. Alan Tuttle and Jean Houston, librarians at the National Humanities Center, deserve special commendation for tracking down a great deal of obscure material for me. I am also grateful to the staff at the Bibliothèque Nationale, Paris; the Bodleian Library, Oxford; the British Library, London; and the Regenstein Library, the University of Chicago. The efforts of the interlibrary loan staff of the D. H. Hill Library at North Carolina State University are especially appreciated.

My greatest debt is to my wife, Doreen, whose love and support have been essential to this project from the beginning. Anything of value here is due in large measure to her.

Abbreviations

WORKS BY HOBBES

Behemoth	*Behemoth; or, The Long Parliament* (Hobbes [1889] 1969b). References are to a dialogue number, followed by a reference to *EW* after a semicolon.
CTH	*The Correspondence of Thomas Hobbes* (Hobbes 1994).
DCo	*Elementorum Philosophiae Sectio Prima De Corpore* (Hobbes 1655). References are to part, chapter, and article numbers separated by periods, followed by a page reference to *OL*; to *EW* when I need to refer to the English version (Hobbes 1656a); or to the original (Hobbes 1655) when it differs from the version anthologized in *OL*.
DH	*Elementorum Philosophiae Sectio Secunda De Homine* (Hobbes 1658). References are to part, chapter, and article numbers separated by periods, followed by a reference to *OL* after a semicolon.
DP	*Dialogus Physicus* (Hobbes 1661b). References are to work as a whole, followed by a reference to *OL* after a semicolon.
EL	*The Elements of Law Natural and Politic* (Hobbes 1969a). References are to part, chapter, and article number separated by periods, followed by a reference to *EW* after a semicolon.
EW	*The English Works of Thomas Hobbes of Malmesbury, now First Collected and Edited by Sir William Molesworth* (Hobbes [1839–45] 1966b).
Examinatio	*Examinatio et Emendatio Mathematicae Hodiernae* (Hobbes 1660). References are to dialogue number, followed by a reference to *OL* after a semicolon.
L	*Leviathan* (Hobbes 1651). References are to part and chapter number separated by a period, followed by a page

number from the 1651 edition; a reference to *EW* follows after a semicolon.

Lux — *Lux Mathematica* ([Hobbes] 1672). References are to chapter number, followed by a reference to *OL* after a semicolon.

MHC — *Mr. Hobbes Considered in his Loyalty, Religion, Reputation and Manners* ([Hobbes] 1662b). References are to the work as a whole, followed by a reference to *EW* after a semicolon.

OL — *Thomae Hobbes Malmesburiensis Opera Philosophica Quae Latine Scripsit Omnia in Unum Corpus Nunc Primum Collecta* (Hobbes [1839–45] 1966a).

PP — *Problemata Physica* (Hobbes 1662a). References are to the work as a whole, followed by a reference to *OL* after a semicolon.

PPAG — *Principia et Problemata Aliquot Geometrica* (Hobbes 1674). References are to chapter number, followed by a reference to *OL* after a semicolon.

PRG — *De Principiis et Ratiocinatione Geometrarum* (Hobbes 1666). References are to chapter, followed by a reference to *OL* after a semicolon.

SL — *Six Lessons to the Professors of the Mathematiques* (Hobbes 1656b). References are to lesson number, followed by a reference to *EW* after a semicolon.

SPP — *Seven Philosophical Problems* (Hobbes 1682). References are to chapter number, followed by a reference to *EW* after a semicolon.

Στίγμαι — *Στίγμαι . . . or Markes of the Absurd Geometry, Rural Language, Scottish Church-Politicks, And Barbarismes of John Wallis* (Hobbes 1657). References are to section number, followed by a reference to *EW* after a semicolon.

WORKS BY WALLIS

AI — *Arithmetica Infinitorum* (Wallis 1656b, part 3). References are to proposition number, followed (where possible) by a reference to *OM* after a semicolon.

Angle of Contact — *De Angulo Contactus et Semicirculi Disquisitio Geometrica* (Wallis 1656b, part 1). References are to chapter number, followed by a reference to *OM* after a semicolon.

Conic Sections	*De Sectionibus Conicis Tractatus* (Wallis 1656b, part 2). References are to chapter number, followed by a reference to *OM* after a semicolon.
Dispunctio	*Hobbiani Puncti Dispunctio; or The Undoing of Mr Hobs's Points* (Wallis 1657b).
Due Correction	*Due Correction for Mr. Hobbes; or Schoole Discipline, for not saying his Lessons right* (Wallis 1656a).
Elenchus	*Elenchus Geometriae Hobbianae* (Wallis 1655).
HHT	*Hobbius Heauton-timorumenos; or A Consideration of Mr. Hobbes his Dialogues in an Epistolary Discourse Addressed to the Honourable Robert Boyle, Esq.* (Wallis 1662).
Mechanica	*Mechanica; sive, De Motu, Tractatus Geometricus* (Wallis 1670). References are to chapter number, followed by a reference to *OM* after a semicolon.
Mens Sobria	*Mens Sobria Seriò Commendata* (Wallis 1657a).
MU	*Mathesis Universalis* (Wallis 1657c, part 2). References are to chapter number, with a citation to *OM* following after a semicolon.
OM	*Opera Mathematica* (Wallis 1693–99).
Treatise of Algebra	*A Treatise of Algebra, Both Historical and Practical* (Wallis 1685). References are to chapter number and page.

WORKS BY OTHER AUTHORS

AT	*Oeuvres de Descartes* (Descartes 1964–76).
Elements	*The Thirteen Books of Euclid's "Elements"* (Euclid [1925] 1956). References are to Book number and proposition, definition, or axiom number.
Exercitatio	*In Thomae Hobbii Philosophiam Exercitatio Epistolica* (Ward 1656).
HOC	*Les Oeuvres Complètes de Christiaan Huygens* (Huygens 1888–1950).
LM	Barrow, *Lectiones Mathematicae* (Barrow 1685). References are to lecture number, with a citation to page number in volume 1 of Barrow (1860) following after a comma.

References to Aristotle use title of work, followed by book and chapter number separated by periods; reference to the Bekker page numbers given marginally in the Revised Oxford Translation of Aristotle (1984) follow after a semicolon.

All translations are by the author unless otherwise indicated in the references section.

CHAPTER ONE

The Mathematical Career of the Monster of Malmesbury

> [T]he doctrine of Right and Wrong, is perpetually disputed, both by the Pen and the Sword: Whereas the doctrine of Lines, and Figures, is not so; because men care not, in that subject what be truth, as a thing that crosses no mans ambition, profit, or lust.
>
> —Hobbes, *Leviathan*

In June of 1645 the English mathematician John Pell wrote to his friend Sir Charles Cavendish seeking assistance in an ongoing controversy with the Danish astronomer-mathematician Christian Severin Longborg, who is better known by his Latinized name Longomontanus. Pell had been appointed professor of mathematics in Amsterdam in 1643 and was known for his work in algebra, and particularly for his lectures on the work of the third-century Greek mathematician Diophantus of Alexandria. The aged Longomontanus, who had once served as an assistant to the venerable Tycho Brahe, had held the professorship of mathematics at Copenhagen since 1607 and over the course of several decades had published a number of supposed solutions to the ancient problem of squaring the circle (Longomontanus 1612, 1627, 1634, 1643, 1644a). Pell saw the last of these, which bore the ponderous title *Rotundi in Plano, seu circuli absoluta mensura, duobus libellis comprehensa*, and promptly penned a two-page refutation (Pell 1644). In a move guaranteed to infuriate Longomontanus, his publisher Johannes Blaeu had the refutation printed and appended to the copies of *Rotundi in Plano* that remained in stock. Longomontanus responded with *Ἐλέγξεος Joannis Pellii contra Christianum S. Longomontanum De Mensura Circuli Ἀνασκευή* in the same year (1644). In this work he rejected the trigonometric principles employed by Pell and reasserted the correctness of his result.[1] Pell was determined

1. The theorem used by Pell asserts that if r is the radius of a circle, a the tangent of an arc less than 45°, and x the tangent of an arc twice as great as arctangent a, then

to pursue the battle and decided to solicit alternative proofs from European mathematicians of the key lemma he had used in his earlier refutation. His intent was to "appeale to y^e^ judgements of all those that by demonstrating my fundamental Theorem . . . can shew themselves able to judge of such a controversy," and he was convinced that "yose ignorant Danes may be so much y^e^ more confounded to see a thing demonstrated so many severall wayes, which Longomontanus sayd was *indemonstrabile*."[2]

Sir Charles Cavendish was the brother of the first earl (and later duke) of Newcastle and a steadfast supporter of King Charles I, but he found himself in France at the time as a result of the English Civil War. While there, he pursued his scientific and mathematical interests by maintaining contacts with European scientific figures, including other Royalist expatriates.[3] Among them was Thomas Hobbes, whose ties to the Cavendish family were long-standing, and whom Sir Charles approached for a proof of the relevant result. Writing in reply to Pell in June of 1645, Cavendish remarked that "I have as you desire procured not onelie the approbation but demonstration of your fundamentall proposition by Mr. Hobbes his meanes. Fermat is not in this towne, and Mersennus is on his waye hither, so that I knowe not whither to write him. But I doute not but more handes with demonstrations might be procured if you desire it."[4] Hobbes's demonstration was eventually published in Pell's *Controversiae de verâ circuli mensurâ . . . Prima pars* (1647), along with contributions by such mathematical luminaries as René Descartes, Bonaventura Cavalieri, and Gil-

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$xr^2 - xa^2 = 2r^2a$. As Isaac Barrow noted in commenting on this exchange, Pell's "most excellent theorem" comes to the same thing as the claim that "the difference between the square of the radius and the square of the given tangent will be in the same ratio to twice the square of the radius, as the given tangent is to the tangent sought" (*LM* 15, 243–44). Longomontanus simply denied this theorem, although his objections are not worth examining here. For a full account of this controversy see Maanen 1986 and Dijksterhuis 1931–32.

2. John Pell to John Leake, 7/17 August 1645; British Library MS. Add. 4280, f. 94^r^.

3. See Jacquot 1949 and 1952c for an account of Sir Charles Cavendish and his connections to the scientific and mathematical work of the seventeenth century. For more on the Cavendish family and its history, see Bickley 1911, especially the fourth chapter, which deals with Newcastle and the English Civil War. For a general study of Royalist expatriates in France during the Civil War see Guitian 1996.

4. Cavendish to Pell, 27 June/6 July 1645; British Library MS. Add. 4278, f. 170^r^. Mersennus is Marin Mersenne, the Minim friar and scientific writer who maintained a voluminous correspondence with nearly all of the important European philosophical and scientific figures of the seventeenth century. Fermat is the great French mathematician Pierre de Fermat.

les Personne de Roberval. These "notable mathematicians" were to serve rather like a jury in hearing the claims of Pell and Longomontanus, and they naturally returned a verdict in Pell's favor.[5] The irony here is quite remarkable: Hobbes, who would later spend years publishing and defending numerous failed attempts to square the circle, published his first mathematical work as part of a campaign to silence an old circle squarer. Indeed, less than a decade after his participation in Pell's battle with Longomontanus, Hobbes would find himself involved in a prolonged and bitter controversy that centered on his claims to have squared the circle, and he would go to his grave insisting that he had solved this ancient geometrical problem.[6]

Near the end of his extraordinarily long and full life, the octogenarian Hobbes penned an account of his accomplishments in the form of a Latin prose vita. In it he recounted a list of his great mathematical achievements and confidently asserted numerous claims to everlasting mathematical glory. This account of his supposed mathematical triumphs is worth considering briefly, as it outlines much that will be of concern in the present book:

> In mathematics, he corrected some principles of geometry. He solved some most difficult problems, which had been sought in vain by the diligent scrutiny of the greatest geometers since the very beginnings of geometry; namely these:

5. Richard Blackbourne's *Vitae Hobbianae Auctarum* reports the episode as follows: "And when the learned Pell publicly attacked the paralogisms of Longomontanus and readily discerned that the whole controversy turned on one theorem that was to be demonstrated, he both demonstrated it after his own style and sought to defend his opinion by inviting demonstrations from the most famous mathematicians in Europe." In evaluation of Hobbes's contributions, he adds "But of Hobbes's reasoning, in order that I not seem to offer immoderate praise through my admiration, I shall only add that it is inferior to none of the others in the brevity of its elegance or the perspicacity of its evidence" (*OL* 1:xxxi–xxxii). The successful outcome of his dispute with Longomontanus seems to have won Pell some recognition; at any rate, Aubrey (repeating the evaluation of Theodore Haak) reports that "his fame was much augmented by his refuting a large book of Longomontanus Quadratura, which caused the Prince of Orange (Henry Fredrick) being about to erect an Academie at Breda, borrowed Mr. Pell from the magistrate of Amsterdam, to grace his new Academy with a man of that fame for a few years" (Aubrey 1898, 2.130).

6. In fact, Hobbes left an incomplete tract on circle quadrature that was apparently his last written work. The fragment dates from the last year of Hobbes's life and was his last attempt to assert his claims to mathematical glory. The dedication contains a revealing statement of Hobbes's attitude toward the rest of the mathematical world at this point. He writes: "And so, after I had given sufficient attention to the problem by different methods, which were not understood by the professors of geometry, I added this newest one" (Chatsworth Library, Hobbes MS. A.9, f.1^r).

> 1. To exhibit a right line equal to the arc of a circle, and a square equal to the area of a circle, and this by various methods. In several books.
>
> 2. To divide an angle in a given ratio.
>
> 3. To find the ratio of a cube to a sphere. In the *Geometrical Problems.*
>
> 4. To find any number of continual mean proportionals between two given lines. In the *Geometrical Problems.*
>
> 5. To describe a regular polygon with any number of sides. In the *Geometrical Rose Garden.*
>
> 6. To find the center of gravity of the quadrant of a circle, and the bilinear figure contained in the arc of a quadrant and its subtense. In the *Geometrial Rose Garden.*
>
> 7. To find the centers of gravity of all types of parabolas *[paraboliformium].* In the book *De Corpore.*
>
> He was the first to construct and demonstrate these, and many other things besides, which (because they will appear in his writings *[legentibus]* and are less important) I pass over. (*OL* 1:xix)

This list is essentially a catalog of Hobbes's mathematical publications and, not coincidentally, a summary of the most eagerly sought mathematical results from antiquity to the early modern era. It is important to observe that Hobbes claims credit for "correcting" some geometric principles as well as solving great outstanding problems of geometry. These claims are not unrelated, for Hobbes was convinced that his reformation of geometry would render essentially any problem solvable. In his view the failures of previous generations of geometers to solve them did not stem from an intrinsic intractability of the problems or from a lack of industry and intelligence on the part of earlier mathematicians. Rather, their failures were the result of a lack of proper method, and Hobbes was convinced that his understanding of the true mathematical method would put him (or anyone else who adopted his principles) in a position to find long-sought solutions to the great problems of classical geometry.

This emphasis upon the importance of proper method recurs throughout Hobbes's writings, whether they concern politics, natural philosophy, or mathematics. The Hobbesian obsession with methodological concerns is also clearly inspired by his appreciation of the deductive structure of geometry. Indeed, it was supposedly the discovery of mathematics at the age of forty that led Hobbes to attempt to cast all of philosophy on the model of geometry. We owe to Aubrey the

famous and revealing account of Hobbes's first appreciation of mathematical method and his subsequent infatuation with the subject:

> He was . . . 40 yeares old before he looked upon geometry; which happened accidentally. Being in a gentleman's library in ———, Euclid's Elements lay open, and 'twas the 47 El. libri I. He read the proposition. 'By G——,' sayd he, 'this is impossible!' So he reads the demonstration of it, which referred him back to such a proposition; which proposition he read. That referred him back to another, which he also read. *Et sic deinceps,* that at last he was convinced of that trueth. This made him in love with geometry. (Aubrey 1898, 1:332)

However embellished this account may be (and it is almost certainly embellished), there is no reason to doubt that Hobbes was deeply impressed with the deductive structure of geometry and especially with the idea that simple and indubitable first principles can yield recondite theorems.[7]

During his prolonged stay in Paris in the 1640s Hobbes was active in the intellectual circle around Marin Mersenne and came in contact with many of Europe's leading scientists, philosophers, and mathematicians. We can gain some idea of the extent of Hobbes's contacts

7. The principal reason for suspecting the complete accuracy of Aubrey's account is the remarkable coincidence that the proposition that supposedly initiated Hobbes into geometry is the Pythagorean theorem: proposition 47 of the first book of Euclid. Moreover, Euclid's proof of the theorem is sufficiently complex to be largely incomprehensible to a complete novice (as Hobbes allegedly was). Pacchi 1968 has drawn attention to a reading list in the library at Chatsworth that lists several hundred books and was marked up in Hobbes's hand sometime between 1625 and 1628. This list seems to have served as a guide to Hobbes's studies in this period and it contains references to a remarkably wide collection of books in mathematics, so it seems unlikely that Hobbes had no exposure to geometry in 1629. Still, there is little doubt that Hobbes undertook intensive mathematical study during his journey to the Continent in that year. In his autobiographical vita Hobbes reports that in that year: "In that journey [to the Continent in the company of Sir Gervase Clifton] he began to read Euclid's *Elements;* and, well-pleased by its method, not because of the theorems, but rather because of its art of reasoning, he read through it most carefully" (*OL* 1:xiv). Although we may be suspicious of points in Aubrey's account, it is not without a ring of truth. For example, his report that "I have heard Mr. Hobbes say, that he was wont to draw lines on his thighs, and on the sheets a-bed" (Aubrey 1898, 1:333) finds an echo in the curiously autobiographical remark in *Leviathan* that "from being long and vehemently attent upon Geometricall Figures, a man shall in the dark, (though awake) have the Images of Lines, and Angles before his eyes" (*L* 1.2, p. 6; *EW* 3:6). Bernhardt (1986) has drawn attention to a passage from the first edition of Hobbes's *Examinatio* (Hobbes 1660, 154) that is strikingly similar to Aubrey's account, but I think that the story remains somewhat exaggerated.

with the Parisian mathematical establishment from Sir William Petty's remark to Pell that "Mr. Hobbes served you in procuring the demonstrations of other French Mathematicians" in the battle against Longomontanus.[8] The intellectual stimulation provided by his extensive contacts with Parisian savants led Hobbes to devote himself to the study of mathematics and natural philosophy, and it is at this time that he set to work drafting a comprehensive summary of the grand philosophical system that he intended to encompass all of mathematics, physics, and political philosophy. These efforts were interrupted by his concern over the course of political events in England and the consequent writing of *Leviathan*, so the grand scheme of a complete system of philosophy remained unfinished by the time of his return to England in 1651. Nevertheless, as the incident with Pell suggests, Hobbes had acquired a considerable reputation in mathematics by the time his masterpiece *Leviathan* was published. On the death of Descartes in 1650, Hobbes's friend Samuel Sorbière could say of the deceased French savant that he was "one of the world's foremost men in algebra and geometry," an opinion which he supported by reference to "Roberval, Bonnel, Hobbes, and Fermat, who are the greatest masters."[9] Word of Hobbes's mathematical prowess no doubt contributed to his being engaged as tutor in mathematics to the prince of Wales (the future Charles II) in 1646, although he reminded Sorbière to "beware of thinking it more important than it is" because "I am only teaching mathematics, not politics" (Hobbes to Sorbière, 24 September/4 October 1646; *CTH* 1:141). In June of 1655 Henry Oldenburg could request his assistance on behalf of an unnamed friend (who may have been Robert Boyle) for advice on "ye best authors, y[t] haue specified those uses [of mathematics]," and ask that Hobbes "would fauour us w[th] y[e] culling out of such authors, and send their names" (Oldenburg to Hobbes, 6/16 June 1655; *CTH* 1:211–12).

Hobbes remarked that the learned community's "expectation of that which should be written by me, was raised partly by the *Cogitata physico-Mathematica* of *Mersennus*, wherein I am often named with honour; and partly by others with whom I then conversed in *Paris*" (*SL* 6, 56; *EW* 7:334). Still, the lofty mathematical reputation he enjoyed in the early 1650s was not based on his publications, which up

8. Petty to Pell, 8/18 November 1645; British Library MS. Add. 4279, f. 172r.

9. Sorbière to Claude de Saumaise, 28 February/10 March 10 1650. Extract printed in Tönnies 1975, 68. Bonnel was a mathematician and doctor from Montpellier who corresponded with Mersenne (*AT* 3:332).

to that point contained essentially no mathematical material.[10] Hobbes intended to cement his reputation as an important mathematician by putting a collection of mathematical results into his treatise *De Corpore*, a volume of first philosophy he had worked on intermittently for years.[11] When it finally appeared in 1655, *De Corpore* was supposed to guarantee Hobbes's place in the mathematical pantheon by (among other things) squaring the circle.

Hobbes's claims to mathematical glory did not go unchallenged, and they form the core of the long and bitter dispute that raged between him and John Wallis, Oxford's Savilian professor of geometry. Wallis was by no means the only critic of Hobbes; his colleague Seth Ward (the Savilian professor of astronomy) published a detailed critique of the Hobbesian philosophy that attacked everything in *De Corpore* that Wallis had passed over (Ward 1656), while many other authors of the period also weighed in with denunciations of Hobbes and his philosophy.[12] Nevertheless, for its length and intensity, the fight between Wallis and Hobbes was the outstanding dispute in Hobbes's sometimes quarrelsome career, and it is not without justice that those who have commented on it characterize the dispute as a war.[13]

10. It is worth observing that mathematical reputations were frequently not linked to published output in Hobbes's day. Such luminaries as Roberval or Viscount Brouncker could establish substantial reputations through correspondence and with little or no published output.

11. The title of the work is *Elementorum Philosophiae Sectio Prima De Corpore*, or in the English translation of 1656 *Elements of Philosophy; The First Section Concerning Body*. Both versions are universally known as *De Corpore*, however, and I will adopt that usage here.

12. See Mintz 1962 and Bowle [1951] 1969 for more on the anti-Hobbes literature. Unfortunately, Mintz ignores the great majority of the Wallis material.

13. In his prose *Vitae Hobbianae Auctarum*, Richard Blackbourne characterizes the course of events this way: "At that time the illustrious Wallis, Savilian professor of geometry at the University of Oxford, sounded the call for this long-running mathematical war by publishing his *Elenchus of Hobbesian Geometry*. This most bitter struggle, fought with compass and rule, and sometimes the exchanged of the most sharp abuse, was carried on by both men and lasted more than twenty years, nor did it finally end until the death of Hobbes" (*OL* 1:xxxviii). Isaac D'Israeli portrayed the controversy as "The Mathematical War between HOBBES and the celebrated Dr WALLIS . . . A series of battles, the renewed campaigns of more than twenty years, can be described by no term less eventful. HOBBES himself considered it as a war, in which he took too much delight. His 'Amata Mathemata' was a war of idle ambition; it became his pride, his pleasure, and his shame. He attempted to maintain his irruption into a province he ought never to have entered in defiance, by a 'new method;' but having invaded the powerful natives, he seems to have almost repented the folly, and retires, leaving 'the unmanageable brutes' to themselves!" ([1814] 1970, 90).

For all the significance that this conflict had for Hobbes personally, it has been little investigated by historians of philosophy or science.[14] To whatever extent there is a received view on the controversy, it is that there is no great point in studying it. Hobbes scholarship generally treats the whole affair as an embarrassment, and historians of mathematics generally disregard the incident, apparently on the grounds that it failed to lead to any interesting mathematical advances. On the few occasions when the controversy is mentioned, it is with an air of puzzlement at the fact that Wallis should have wasted so much time and effort on an opponent as unworthy as Hobbes.[15] In his study of Wallis's mathematical career, J. F. Scott summed up what remains the prevailing attitude of bewilderment at the duration and intensity of this quarrel. After remarking upon the "strange lack of restraint on the part of each of the disputants," Scott declares:

> For nearly a quarter of a century the two disputants had waged a contest which shed a lustre round neither of them, and one cannot help wondering why Wallis should have been so eager to expose the mathematical short-comings of one so ill-equipped as Hobbes. For apart from his published exposures, Wallis disparages, and in no unmeasured language, the claims of his antagonist in more than a score of lengthy letters to different members of the Royal Society. Hobbes' claims were so preposterous that a wiser than Wallis would have left him to work out his own destruction. (Scott 1938, 170)

This received view is plausible if we assign importance to Hobbes's mathematical work in proportion to his contributions to the advancement of seventeenth-century mathematics. But it is not obvious that

14. The dispute, and Hobbes's mathematical writings generally, have not been completely ignored. Older works that summarize the dispute without going into the technical details include Cajori 1929 and Scott 1938, chap. 10. A recent important study of the dispute is Probst 1997. Studies of Hobbes's philosophy of mathematics that do not concentrate on the dispute with Wallis include Bird 1996; Breidert 1979; Giorello 1990; Grant 1990; Grant 1996; Jesseph 1993a; Jesseph 1993b; Keller 1992; Pycior 1987; Pycior 1997, chap. 6; Sacksteder 1980; Sacksteder 1981a; Schumann 1985; and Weinreich 1911, pt. 2.

15. Thus, Richard Peters concludes his brief summary of the dispute with the remark that "[e]ven when allowance has been made for the fact that squaring the circle did not then seem quite such a preposterous project as it does now, it remains evident that Hobbes' sublime confidence in his own ability led him to make rather a pathetic exhibition of himself. Wallis had probably forgotten more mathematics than Hobbes ever knew. . . . Nevertheless Wallis was not an attractive man and his brilliant demolition of Hobbes' argument was interspersed with pontifical and boorish invective. A greater

this criterion of importance is correct. We can grant that Hobbes failed to prove anything of significance without thereby being committed to the view that his mathematical writings do not merit scholarly attention. Whatever the caliber of his mathematical work, Hobbes assigned great importance to mathematics, and an understanding of his views on the subject must be part of our understanding of his philosophy as a whole. Indeed, Hobbes's insistence on the importance of proper method and his lavish praise of mathematical method invite scrutiny of his mathematical work to see how closely his geometric practice follows his methodological theory.

More importantly, the Hobbesian mathematical corpus does not consist exclusively of attempts to square the circle. Much of what Hobbes wrote on mathematics concerns philosophical and methodological issues independent of his failed quadratures, and these writings can provide a valuable insight into seventeenth-century philosophy and mathematics. To understand Hobbes's philosophy in any depth, it is necessary to understand the place of mathematics in his grand system. Furthermore, to appreciate the groundbreaking mathematical developments of the seventeenth century, we need the perspective of those, like Hobbes, who resisted them. My purpose here is therefore not to rehabilitate Hobbes's mathematical reputation, but rather to focus on his mathematical work as a way of improving our understanding of his philosophy and the context in which it developed. Such an inquiry will necessarily lead into territory not generally covered in a study of Hobbes and his philosophy, but it is an inquiry worth undertaking for anyone interested in the Hobbesian philosophical enterprise and its reception. It also provides a chance to test some historiographic theses, and particularly the claim that the history of philosophy, science, or mathematics is driven exclusively by social and political factors.

Although I will be primarily concerned with the exchanges between Hobbes and Wallis, there were many other figures who contested Hobbes's mathematical claims either in print or in correspondence, and it will be necessary to bring some of this additional material into consideration at various points in this study. Hobbes's reputation was great enough in 1655 that his mathematical works were taken seriously in English and Continental circles, and in its early stages his dispute with Wallis seems to have attracted a good deal of attention from

man could have afforded to be kinder to the old gentleman in spite of such pretentious provocation" (Peters 1956, 40).

the learned world.[16] By the mid 1660s, however, Hobbes had lost essentially all credibility with the mathematical public and his works were largely disregarded. In any case, although this point is occasionally overlooked in the literature, it is worth remembering that Wallis was not the only person who read and critiqued Hobbes's mathematical work.[17]

In the course of their dispute Hobbes and Wallis covered issues that went well beyond mathematical and methodological concerns. Questions of political loyalty, church government, theology, and classical philology were all raised and debated as the two traded vituperative pamphlets. The length of the dispute and the variety of topics concerned in it make an exhaustive treatment of the issues impossible, but I intend to cover as much of the relevant territory as is possible within the confines of a single book. The rather tangled history of this protracted conflict can best begin with an overview of the exchanges and an introduction to the mathematical background, both of which will be my concern in this chapter. The first section sketches an overview of the Hobbes-Wallis controversy, while the second introduces the necessary mathematical history.

1.1 THE DISPUTE IN OVERVIEW

The roots of the conflict are complex and will be examined in detail in chapter 2, but the controversy between Hobbes and Wallis began in earnest in 1655 with the publication of Hobbes's treatise *De Corpore,* in which he claimed that his methodological principles had enabled

16. Not all observers quite understood the nature of the dispute at the outset, however. The Parisian mathematician Claude Mylon, commenting to Hobbes's friend François du Verdus, remarked that "I know that the universities do not approve too highly of his work, and I believe that the geometer Dr John Wallis, Savilian Professor in the University of Oxford, has used Aristotle to attack it" (du Verdus to Hobbes 20 February/1 March 1656; *CTH* 1:239). Mylon was one of many who critiqued the mathematics of *De Corpore,* and we will be concerned with his efforts in chapter 6.

17. Bird (1996, 218) mistakenly represents Wallis as nearly the only person to have taken notice of Hobbes's mathematics when he claims that "Wallis distinguished himself as the only mathematician to have taken Hobbes's geometry seriously. Only three others—the Belgian philosopher Moranus, John Pell, sometime professor at Amsterdam and Breda, and Viscount Brouncker—are known to have given it more than passing thought." This is seriously in error, first because the mathematical sections of Moranus's critique of *De Corpore* were written by André Tacquet, and more importantly because Bird fails to mention Christiaan Huygens, François de Sluse, Claude Mylon, Seth Ward, Laurence Rooke, Roberval, and Pierre de Carcavi, all of whom gave Hobbes's geometry at least as much attention as Brouncker or Pell. This will become evident in chapter 6.

him to square the circle, rectify curvilinear arcs, and solve other outstanding geometrical problems. These claims were accompanied by what appear to be proofs of some of the most eagerly sought geometrical results of the seventeenth century. Within the year Wallis responded with his *Elenchus Geometriae Hobbianae*, in which he attacked the entire account of geometry contained in *De Corpore* and pointed out numerous technical errors in Hobbes's attempted solution to such problems as the squaring the circle.[18] Hobbes replied in an appendix to the English version of *De Corpore* entitled *Six Lessons to the Professors of the Mathematiques* (1656). The professors concerned were Wallis and Ward—holders of the Savilian Chairs of Geometry and Astronomy, respectively. Ward had attacked Hobbes in his 1656 *In Thomae Hobii Philosophiam Exercitatio Epistolica*, which undertook the refutation of Hobbes's natural, moral, and political philosophy, while adding scattered references to inadequacies in his mathematics. Although the *Six Lessons* were addressed to both Ward and Wallis, they are almost exclusively concerned with replying to Wallis's *Elenchus* and going over to the attack against other of his writings. The rhetorical demands of this undertaking led Hobbes to criticize some of Wallis's Latin usage, as well as to urge purely mathematical objections to his *Arithmetica Infinitorum* of 1656, which was an influential and highly regarded text that employed a mathematical technique known as the "method of indivisibles."

Wallis did not wait long to answer and kept up the high level of invective with his *Due Correction for Mr. Hobbes; or Schoole Discipline, for not saying his Lessons right* (1656). In the following year, Hobbes responded with his *Στίγμαι Αγεωμετρίας, Αγροικίας, Αντιπολιτείας, Αμαθείας; or, Markes of the Absurd Geometry, Rural Language, Scottish Church-Politicks, And Barbarismes of John Wallis Professor of Geometry and Doctor of Divinity* (1657). As is evident from the title, this work launched an offensive against Wallis on all fronts—mathematical, grammatical, political, and personal. While preparing his response, Hobbes entered into a correspondence with Henry Stubbe, a fellow of Oxford's Christ Church College whose classical learning proved useful in addressing the philological points that had come into dispute. Stubbe had undertaken the translation of Hobbes's *Leviathan* into Latin (a task that remained unfinished) and

18. Among other things, Wallis noted that *De Corpore* had apparently been snatched from the press and undergone (unsuccessful) correction of some key mathematical sections. The tangled publishing history of *De Corpore* will be taken up at the end of chapter 3.

was deeply involved in theological and political struggles against Wallis at Oxford, but he also found time to act as an agent for Hobbes—reporting Wallis's plans and attempting to rally support for Hobbes within the university community.[19] Hobbes appended an extract from a letter from Stubbe to *Στίγμαι* in which Stubbe defended Hobbes on certain finer points of Latin usage and Greek etymology while ridiculing Wallis's efforts at philological criticism. Wallis did not wait long to answer, and his reply took the form of a book entitled *Hobbiani Puncti Dispunctio; or the Undoing of Mr. Hobs's Points* (1657). This drew no direct rebuttal from Hobbes, but Stubbe responded with *Clamor, Rixa, Joci, Mendacia, Furta, Cachinny; or, a Severe Enquiry into the late Oneirocritica Published by John Wallis* (1657).

This began a three-year period of calm, but an ominous chord was struck with Wallis's publication of *Mathesis Universalis* (1657). This specimen of "universal mathematics" is a wide-ranging work that purports to expound the true nature of all mathematics (including its historical development) while also presenting the essentials of the major branches of the subject. Hobbes is briefly and contemptuously mentioned in chapter 24 as one of those who has tried and failed to square the circle, but Wallis undertakes no extended critique of his work, except to mention that the publication of *Mathesis Universalis* was delayed by the necessity of attacking Hobbes's mathematical publications.[20]

19. See Jacob 1983 (chap. 1) and Malcolm's biographical register (*CTH* 2:899–902) for accounts of Stubbe and his activities. I will investigate the Stubbe-Hobbes connection more closely in chapters 2 and 7. It is worth mentioning that Stubbe attacked Wallis and his associates on other occasions that had nothing to do with Hobbes. He published *The Savilian Professours Case Stated* (1658) as a censure of Wallis's appointment as Oxford's keeper of the archives, and later went on to write a number of denunciations of the Royal Society (Stubbe 1670a, 1670b, 1670c, 1671). These latter were evidently written at the behest of Dr. Baldwin Hamey of the Royal College of Physicians, who feared that the Royal Society would encroach upon the claims of the College of Physicians (Westfall 1958, 23). This last attack seems to have displeased Hobbes, as Aubrey reports that he "much esteemed [Stubbe] for his great learning and parts, but at latter end Mr. Hobbs differ'd with him for that he wrote against the lord chancellor Bacon, and the Royall Societie" (Aubrey 1898, 1:371).

20. Wallis notes that the search for an exact value of π (upon which the quadrature of the circle depends) has remained unsuccessful. "Which [exact value] when they imagine themselves (after all the others) to have found it, *Joseph Scaliger, Severinus Longomontanus*, and most recently *Thomas Hobbes*, in dreaming the immortal praises thence due to themselves alone, are merely hallucinating" (*MU* 24; *OM* 1:132). Of Hobbes's role in delaying the publication of the *Mathesis Universalis*, Wallis claims that "But in the meantime (aside from publishing some theological writings) it happened that *Hobbes* was twice to be castigated, now for his geometrical incompetence [*ἀγεωμετρη-*

Hobbes's reaction to the *Mathesis Universalis* was not immediate, but in 1660 he published six dialogues under the title *Examinatio et Emendatio Mathematicae Hodiernae*. The first four dialogues are an extended commentary on and rebuttal to Wallis's *Mathesis Universalis,* while the fifth attacks other works (principally the *Arithmetica Infinitorum*), and the sixth presents sixty-eight propositions concerning the quadrature of the circle and numerous other geometrical problems. Wallis exercised uncharacteristic restraint by not publishing a reply, although he was certainly not impressed by Hobbes's efforts.[21]

Hobbes renewed his claims to preeminence in the mathematical world the next year (1661) with an anonymous solution to the ancient problem of doubling the cube, published in Paris under the title *La Duplication du Cube par V.A.Q.R.* Wallis recognized the author of the work and responded immediately with a letter to an unnamed gentleman (possibly Viscount Brouncker) exposing the error in the alleged solution.[22] Not to be deterred, Hobbes appended a Latin version of this cube duplication to the 1661 criticism of Robert Boyle and the Royal Society that he entitled *Dialogus Physicus, sive De Natura Aeris.* Wallis responded with a severe critique of all things Hobbesian in his *Hobbius Heauton-timorumenos; or A Consideration of Mr. Hobbes his Dialogues in An Epistolary Discourse Addressed to the Honourable Robert Boyle, Esq.* (1662). This attack on Hobbes takes principal

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σίαν] in *De Corpore,* and now for the railing and abuse [λοιδορήματα] with which he has attacked all of the schools and academies, both ancient and recent. He is puffed up with nothing, and offering for sale his ignorance with this open contempt of everything" (*MU Dedicatio; OM* 1:15–16). The "theological writings" Wallis mentions here were published in the 1657 collection *Mens Sobria,* which will concern us in chapter 7.

21. In a summary review of Hobbes's mathematical publications from 1669, Wallis remarks that "I judged that these dialogues should be ignored, considering that so far as they concerned me, they barely contained anything other than the *Six Lessons,* set forth in a different form, to which I had already responded. Nor then did he seem a worthy adversary, who gravely opposed himself to all mathematicians" (Wallis 1669a, 21). It is worth noting that the version of the *Examinatio* in the 1668 Amsterdam edition of Hobbes's *Opera* differs significantly from the original by altering much of the material in the sixth dialogue.

22. The letter survives as no. 51 in the Chatsworth collection of Hobbes's correspondence and is dated 23 June/3 July 1661. Malcolm (*CTH* 1:xlvi n. 5) suggests that "the most likely recipient was Lord Brouncker" because Wallis ends the letter with the closing "Dominationis vestrae observantissimus" ("your Lordship's most dutiful servant"), which would be a very odd way to address Hobbes. In the letter, Wallis reports that the sheet containing the duplication of the cube had been brought to him the previous evening and then embarks on a close examination of Hobbes's argument and reveals the false supposition upon which it is based. This episode will be examined in more detail in chapter 6.

aim at the *Examinatio,* but it also contains an encyclopedic summary of his various attempted quadratures and showed the tortured history of the Hobbesian mathematical enterprise.[23]

Hobbes soldiered on in pursuit of mathematical glory and continued to circulate cube duplications, including one that he submitted to the Royal Society in September of 1662 (with the approval of Charles II), evidently with the purpose of gaining admission to the society by having his work approved by the fellows at the request of their royal patron. Hobbes also appended sixteen propositions on the quadrature of the circle and a revised cube duplication to the exposition of his physics in the *Problemata Physica* of 1662; this work also contained an appeal to Continental mathematicians to examine the algebraic techniques used by Wallis and accepted by English mathematicians as refutations of Hobbes's efforts. In the same year, Hobbes replied to *Hobbius Heauton-timorumenos* with *Mr. Hobbes Considered in his Loyalty, Religion, Reputation, and Manners.* This publication attempted to repair some of the damage done to Hobbes's reputation and abandoned the mathematical issues in dispute for questions of political loyalty and religious orthodoxy. Its appearance marked the end of the second phase of the dispute, but it was not the final chapter.

In 1666 Hobbes weighed in with *De Principiis et Ratiocinatione Geometrarum . . . Contra fastum Professorum Geometriae.* The aim of this work was to restate the Hobbesian principles of geometry and critique the traditional understanding of the subject. Wallis responded with a scathing review of the piece in the *Philosophical Transactions of the Royal Society* in August of 1666. Hobbes's *Opera Philosophica* of 1668 contained some revised versions of his mathematical works, which were followed in 1669 by the publication of *Quadratura Circuli, Cubatio Sphaerae, Duplicatio Cubi, Breviter demonstrata.* Wallis immediately refuted the entire work in his *Thomae Hobbes Quadratura Circuli, Cubatio Sphaerae, Duplicatio Cubi; Confutata* (1669). With lightning speed, Hobbes produced an amended second edition of the work in 1669, in which he responded to Wallis's criticisms. Wallis obliged with a second edition of his refutation, still in 1669, to which Hobbes made no direct response.

In 1671 Hobbes petitioned the Royal Society to hear his claims against Wallis and presented three papers attacking Wallis's mathematical work. These were answered by Wallis in the Royal Society's *Philo-*

23. See *HHT* (104–20) for a remarkable summary of the dozen Hobbesian quadratures that preceded the publication of the *Examinatio.*

sophical Transactions of 8/18 September 1671, even though Hobbes's original papers were not published in the *Transactions.* Hobbes reacted in the same year by publishing *Three Papers Presented to the Royal Society Against Dr. Wallis; Together with Considerations on Dr. Wallis his Answer to them.* In the meantime Hobbes published an anonymous collection of his alleged results and a further attack on Wallis under the title *Rosetum Geometricum; Sive Propositiones Aliquot Frustra antehac tentatae; Cum Censura brevi Doctrinae Wallisianae de Motu.* The "Doctrine of Motion" attacked in this piece was Wallis's *Mechanica; sive, De Motu, Tractatus Geometricus* (1670), and much of Hobbes's critique concerns Wallis's use of infinitesimal methods in his presentation of mechanics along with a restatement of other points from the controversy. Wallis replied in the *Philosophical Transactions of the Royal Society* for 19/29 June and 17/27 July of 1671.

Not to be silenced, Hobbes published a review of the various controversial issues debated between himself and Wallis under the title *Lux Mathematica, Excussa Collisionibus Johannis Wallisii . . . et thomae Hobbesii.* This anonymous work was addressed to the Royal Society and pretended to be an impartial review of the major points of contention between the two disputants from 1655 to 1671. Not surprisingly, *Lux Mathematica* declares Hobbes the winner of every contested point. Wallis again confined his reply to denunciation of the piece in the *Philosophical Transactions* for 19/29 August and 14/24 October 1672.

Two years elapsed before Hobbes ventured to publish anything further regarding mathematics or Wallis, but in 1674 (at the age of eighty-six) he brought out a collection of mathematical results in *Principia et Problemata Aliquot Geometrica Antè Desperata, Nunc breviter Explicata et Demonstrata.* This was not directly attacked by Wallis, and its publication marks the beginning of the end of the Hobbes-Wallis dispute. The final episode occurred in 1678 when Hobbes published his last work, *Decameron Physiologicum: Or, Ten Dialogues of Natural Philosophy . . . To which is added the Proportion of a straight Line to half the Arc of a Quadrant.* These ten dialogues contain an exposition of Hobbesian physics and a critique of Wallis's *Mechanica,* while the appended mathematical paper is Hobbes's attempt to rectify circular arcs. Wallis did not bother to reply to this work, and Hobbes's death in the following year closed the dispute. But even as he went to his grave, Hobbes refused to abandon the fight. Among Hobbes's manuscripts is an unfinished tract on circle quadrature dating from the last year of Hobbes's life (Library, Chatsworth House, Hobbes Manu-

scripts, MS A.9), and it is easy to imagine that his last intellectual efforts were devoted to the unrepentant assertion of his geometrical claims.

1.2 THE MATHEMATICAL BACKGROUND

The problems Hobbes claimed to have solved derive from the tradition of classical geometry and were standard fare in the mathematical literature of the early modern era. Although these were familiar in Hobbes's day, they are less likely to be well understood by a contemporary reader, and it is therefore necessary to spend some time exploring the mathematical context in which the Hobbes-Wallis controversy took place. This task requires a brief account of the classical conception of mathematics and an introduction to the great geometric problems from antiquity: the quadrature of the circle, the duplication of the cube, and the trisection of the angle. After setting out this part of the relevant background, I turn to a short exposition of the two most important mathematical methods of the seventeenth century: the analytic geometry of Descartes and the method of indivisibles pioneered by the Italian mathematician Bonaventura Cavalieri.

1.2.1 The Classical Conception of Mathematics

Philosophical discussions of mathematics in the seventeenth century were largely informed by the doctrines inherited from ancient Greek sources. Although there was by no means unanimity of opinion among classical authors on all points, a broadly shared classical approach to the philosophy of mathematics was still influential in the early modern period. According to the classical point of view there are two fundamentally distinct branches of mathematics: arithmetic and geometry. The former takes numbers as its object and is concerned with discrete quantities that can be expressed as collections of units. The latter deals with continuous and infinitely divisible magnitudes such as lines, figures, and angles.

Further distinctions and refinements are possible within this general classification of the mathematical sciences. Proclus, the fifth-century neo-Platonist whose commentary on the first book of Euclid's *Elements* is an essential source for the traditional philosophy of mathematics,[24] reports that

24. On Proclus and his relationship to philosophy and science, see Siorvanes 1996.

> [t]he Pythagoreans considered all mathematical science to be divided into four parts: one half they marked off as concerned with quantity *(ποσόν)*, the other half with magnitude *(πηλίκον)*; and each of these they posited as twofold. A quantity can be considered in regard to its character by itself or in its relation to another quantity, magnitudes as either stationary or in motion. Arithmetic, then, studies quantity as such, music the relations between quantities, geometry magnitude at rest, spherics [i.e., astronomy] magnitude inherently moving. (Proclus 1970, 29–30)

Others, such as the first-century mathematical encyclopedist Geminus, opted for a slightly more complex and exhaustive division of the mathematical sciences: Geminus first distinguished arithmetic and geometry as the pure sciences concerned with intelligible objects not apprehended by the senses, and then distinguished six applied mathematical sciences (mechanics, astronomy, optics, geodesy, canonics, and calculation) that treat of sensible objects in accordance with the principles of pure mathematics.[25] Although such classificatory schemes were not universally accepted in their entirety, the fundamental distinction between arithmetic and geometry was not seriously challenged in the classical period.

Because Hobbes's mathematical work is devoted almost exclusively to geometry, we can confine our investigation to the classical philosophy of geometry, especially as it is expressed in the *Elements* of Euclid. The Euclidean presentation begins with three kinds of elementary principles or starting points: axioms, definitions, and postulates. In accordance with Aristotle's discussion of demonstrative science in the *Posterior Analytics*, axioms (or "common notions") apply to any science and include such principles as "Things which are equal to the same thing are also equal to one another" (*Elements* 1, ax. 1) or "If equals be subtracted from equals, the remainders are equal" (*Elements* 1, ax. 3). Definitions are principles specific to a given science and state the fundamental or essential properties of its objects, such as the definition of a plane angle as "the inclination to one another of two lines

25. As Proclus reports: "[O]thers, like Geminus, think that mathematics should be divided differently [than the Pythagoreans]; they think of one part as concerned with intelligibles only and of another as working with perceptibles and in contact with them. By intelligibles, of course, they mean those objects that the soul arouses by herself and contemplates in separation from embodied forms. Of the mathematics that deals with intelligibles they posit arithmetic and geometry as the two primary and most authentic parts, while the mathematics that attends to sensibles contains six sciences: mechanics, astronomy, optics, geodesy, canonics, and calculation" (Proclus 1970, 31).

in a plane which meet one another and do not lie in a straight line" (*Elements* 1, def. 8). Finally, postulates are demands that a certain construction be admitted or effected, such as the postulate "To describe a circle with any centre and distance" (*Elements* 1, postulate 3), which permits the construction of a circle of any radius from any point in a plane.

According to the Aristotelian methodology, all sciences must take certain fundamental principles for granted, and the development of the science involves deduction from these basic principles. No science, geometry included, can prove the existence of its proper object, but instead proceeds from the assumption that its object exists. As Aristotle explains:

> I call the principles in each genus those which it is not possible to prove to be. Now both what the primitives and what the things dependent on them signify is assumed; but that they are must be assumed for the principles and proved for the rest—e.g. we must assume what a unit or what straight and triangle signify, and that the unit and the magnitude are; but we must prove that the others are. (*Posterior Analytics* 1.10; 76a 31–36)

Thus, geometry assumes the existence of points, lines, and surfaces, as well as the meaning of its fundamental terms, but it does not attempt to show that things answering to its basic definitions exist. Similarly, the geometric postulates are unchallenged "licenses" to effect constructions answering to specified conditions. But within the framework of these principles, all other geometric propositions are to be derived by a chain of rigorous logical deduction, which establishes the remaining attributes of geometric objects as firmly as the first principles themselves.

The Euclidean *Elements* were generally taken as the prime example of a properly developed science conforming to the Aristotelian canons of demonstration. Indeed, this is largely the reason that medieval thinkers were interested in geometry at all, since the geometry of Euclid was the best-developed example of a properly Aristotelian science.[26] There are notorious shortcomings in the Euclidean presentation

26. As Mahoney observes, "[T]he primary purpose for learning geometry in the Middle Ages was not to carry out further research in the area, but, rather, to understand the geometrical references of Aristotle and the Church Fathers or to be able to do the mathematics demanded by astronomy or optics or to improve mensurational practice and the instruments designed for it" (Mahoney 1978, 153). In commenting on the edi-

of geometry: key principles go unstated, some proofs contain logical gaps, and the famous "parallel postulate" (*Elements* 1, postulate 5) fails to have the simplicity and self-evidence commonly demanded of geometric first principles. The "official" distinction between postulates, axioms, and definitions is also only poorly observed in Euclid: not all of the postulates are really licenses to effect constructions; the fourth postulate of the first book, for example, simply declares that "[a]ll right angles are equal to one another" (*Elements* 1, postulate 4). In fact, much of the history of geometry after Euclid can be read as the search for improvements designed to bring geometry up to the Aristotelian standard by reorganizing its first principles and filling in apparent gaps in the deductive structure.[27] Sir Henry Savile, whose bequest endowed the Savilian Professorships of Mathematics at Oxford, remarked in his commentary on the first book of the *Elements* that "postulates and axioms have this in common, that they require no demonstration, and need be proved by no argument but are taken as manifest, and they are thereafter the principles of all the other things that follow" (1621, 131). But because the fifth postulate of book 1 and the last definition of book 6 seem to lack this kind of self-evidence, Savile regarded them as the only two "flaws or blemishes on the most beautiful body of geometry," and he left to his successors the task of proving them from the remaining principles (Savile 1621, 140–41).[28] Hobbes was convinced that his own work provided the basis for a genuine science of geometry, and he plainly regarded himself as the man who

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tions of Euclid developed by Adelard of Bath and Campanus of Novara, Mahoney notes that "[i]ndeed, it was the argument rather than the mathematical content that seems most to have interested Adelard and Campanus. For the *Elements* was accompanied into scholastic thought by Aristotle's *Posterior Analytics*, which held it up as a model of scientifically demonstrated knowledge" (1978, 154).

27. See Whiteside 1960–62, pt. 1, and Mancosu 1996, chap. 1, for more extensive accounts of seventeenth-century investigations into geometry and Aristotelian methodological principles. Mueller (1981) investigates Euclidean proof structures with modern logical techniques. Dear (1995b) gives an account of the mathematical model of knowledge in Hobbes's day, especially as understood in philosophy.

28. As I mentioned, the fifth postulate of book 1 is the notorious parallel postulate. It asserts "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." Aside from not actually being a license to construct anything, the postulate seems too convoluted to pass the test of intuitive self-evidence required of postulates. The last definition of book 6 is actually spurious. It declares that "[a] ratio is said to be compounded of ratios when the sizes of the ratios multiplied together make some (?ratio, or size)" (*Elements* 1, def. 5). We will be concerned with this definition and its interpretations in chapter 4.

could raise geometry to the standard recognized by Aristotle, Savile, and the tradition.[29]

Propositions proved in Euclidean fashion were commonly distinguished into two groups: theorems and problems. A theorem is a claim to the effect that a certain class of geometric objects has a specific property, such as "[i]n any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles" (*Elements* 1, prop. 32). A problem, in contrast, shows how to construct a geometric object answering a certain description. Thus, "[t]o draw a straight line at right angles to a given straight line from a given point on it" (*Elements* 1, prop. 11) is a problem that begins with certain stated conditions (a given straight line and a point on the line), and then shows how to construct a perpendicular from the given point. Proclus marks the distinction between theorems and problems in essentially this way, and claims that Euclid himself observed the distinction:

> Again the propositions that follow from the first principles [Euclid] divides into problems and theorems, the former including the construction of figures, the division of them into sections, subtractions from and additions to them, and in general the characters that result from such procedures, and the latter concerned with demonstrating inherent properties belonging to each figure. (Proclus 1970, 63)

Christopher Clavius, the great Jesuit mathematician of the late sixteenth and early seventeenth centuries,[30] illustrated this classical distinction by comparing mathematics and dialectics:

> All demonstrations of mathematicians are divided by ancient writers into problems and theorems. A demonstration that demands that something be constructed and teaches how to construct it they call a *problem*. For if someone desires to show that upon a finite right line an equilateral triangle can be constructed, a demonstration of this sort will be called a problem, because it teaches how an equilateral triangle may be constructed upon a

29. In fact, he claims in the epistle to the *Six Lessons* that in chapters 7 through 13 of *De Corpore* that "I have rectified and explained the Principles of the Science [of geometry], *id est*, I have done that business for which Dr. *Wallis* receives the wages" (*SL* epistle; *EW* 7:185).

30. On Clavius and his treatment of mathematics see Dear 1995a, chap. 2; Lattis 1994, chap. 2; and Mancosu 1996, chap. 1.

> finite right line. This type of demonstration is called a problem for its similitude to the problems of dialectics. Just as dialecticians call that question, each of whose contradictory parts is credible *[probabile]* (of which sort is the question whether the whole is really distinguished from its parts taken together) a problem, so also that which is sought by mathematicians when they demand that something be constructed, the contrary of which can also be effected, is called a problem. For example, if someone proposes to himself to demonstrate that upon a finite right line an equilateral triangle can be constructed, he poses a problem because both a nonequilateral triangle (namely isosceles or scalene) can be constructed upon the same line. . . . But they call that demonstration that examines only some aspect *[passio]* or property of one or several quantities at once a *theorem.* So if someone desires to demonstrate that in every triangle the sum of the three angles is equal to two right angles, they will call such a demonstration a theorem, because it does not demand or teach to construct a triangle or anything else, but solely contemplates this aspect of any constructed triangle, namely that the sum of its angles is equal to two right angles. Whence from this contemplation the demonstration is called a theorem. In a theorem it can by no means be made to happen that both parts of a contradiction are true. For if someone demonstrates that the sum of all angles of any triangle is equal to two right angles, it could by no means happen that they are also unequal to two right angles. And the same is to be understood in other theorems. And so to sum it up, that which teaches to construct something mathematical that is sought, but whose opposite can be effected, is a problem; but that which teaches to construct nothing, and whose opposite part remains perpetually false, is called a theorem. (Clavius 1612, 1:9)

This somewhat muddled distinction between problems and theorems was not always taken too seriously, since any problem can be rephrased as a theorem stating the possibility of constructing a certain kind of geometrical object.[31] Nevertheless, the general nature of geometric problems should be clear: they are constructions to be effected on the basis of first principles. Three problems in particular captured the

31. Isaac Barrow, for instance, draws attention to this fact when he notes that "every possible problem can be taken for a theorem, in so far as this possibility is demonstrable" (*LM* 6, 99). He consequently places little weight on the distinction.

imagination of geometers from antiquity, and it is to a consideration of them that I now turn.

1.2.2 *The Three Great Classical Problems*

To construct a square equal in area to a given circle, to divide any given angle into three equal sections, and to construct a cube double in area to a given cube: these are the three great geometric problems from classical antiquity. The long search for solutions to them led to the development of powerful new mathematical tools as well as innumerable wasted hours of intellectual effort. It is now widely known that the problems are unsolvable, but it is important to clarify the nature of the problems and to explain in a general way why they are unsolvable. In Hobbes's day it was an open question whether these problems could be solved, although the repeated failure to find appropriate solutions did provide a certain amount of "inductive" evidence that they could not be conquered. In fact, by the middle of the seventeenth century many had concluded that the problems (and particularly that of squaring the circle) could not be solved with the resources of classical geometry.

1.2.2.1 THE QUADRATURE OF THE CIRCLE The origins of the problem of squaring the circle are obscure, but it seems to have developed naturally out of simpler problems that involve finding the area of rectilinear figures. The resources of Euclidean geometry easily allow the construction of a square equal in area to any rectilinear figure. Indeed one of Euclid's most elementary results is the solution to the problem "[t]o construct a square equal in area to a given rectilinear figure" (*Elements* 2, prop. 14), and it is natural to think of the quadrature of the circle as an attempt to extend this result to the simplest of the curvilinear figures. One of the key results needed to prove *Elements* 2, prop. 14, is *Elements* 1, prop. 45, the problem "[t]o construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure." Proclus, in commenting upon this proposition, reports that

> [i]t is my opinion that this problem is what led the ancients to attempt the squaring of the circle. For if a parallelogram can be found equal to any rectilinear figure, it is worth inquiring whether it is not possible to prove that a rectilinear figure is equal to a circular area. Indeed Archimedes proved that a circle is equal to a right-angled triangle when its radius is equal to one of the

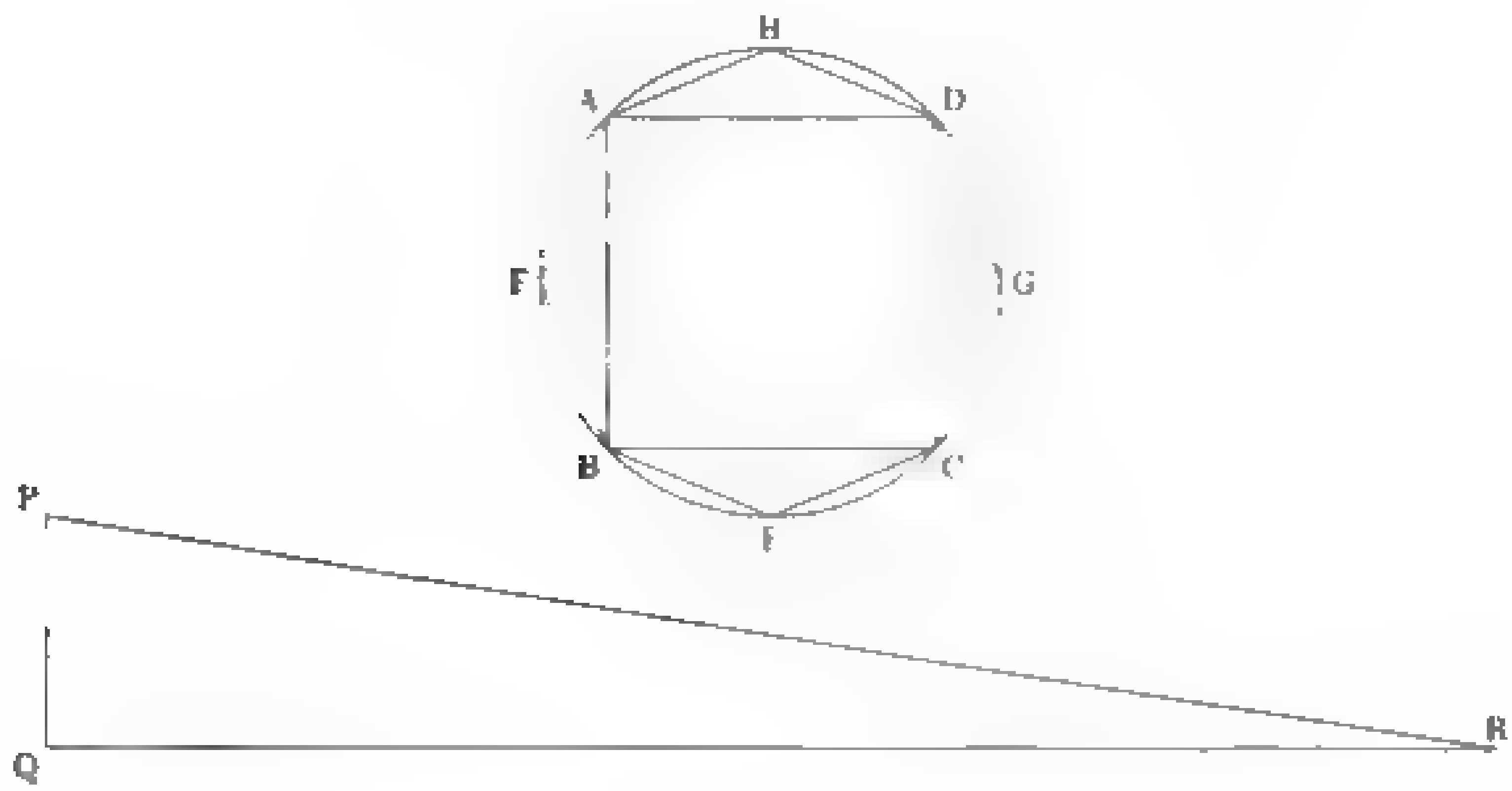

Figure 1.1

sides about the right angle and its perimeter is equal to the base. But of this elsewhere. (Proclus 1970, 335)

Whatever the truth about the origins of the problem, it is clear what it involves and what kind of solution would be required: given a circle of radius *r*, we need to construct a square equal in area to it, using only the means available in Euclid's geometry. The general problem of quadrature is that of finding a square equal in area to any given figure, but it is the quadrature of the circle that is the most important case of this general problem.[32]

Proclus's allusion to the Archimedean result concerning the equality of a circle and a right triangle is instructive for understanding the problem of squaring the circle, because it underscores what would be required for a complete solution to the problem. Proposition 1 of Archimedes' treatise *On the Measurement of the Circle* asserts that "[t]he area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle" (Archimedes 1912, 91). The proof of this theorem employs a classical technique known as the "method of

32. Grattan-Guinness has pointed out that Euclid actually observed a strict division between curvilinear and rectilinear regions, and does not even raise the question of whether the area of a circle and a square can be compared. "[Euclid] does not equate a rectilinear region with a curvilinear one; indeed, in connection with the famous problem of squaring the circle, his commentator Proclus . . . explicitly mentioned this possibility as a worthwhile research topic. Euclid may well have deemed this problem, and similar ones such as squaring lunes, as not Element-ary—and with good justice!" (Grattan-Guinness 1996, 365).

exhaustion," which uses sequences of approximations to determine the value of a sought geometric quantity. In this particular case, Archimedes begins with the given circle *ABCD* and the right triangle *PQR*, such that the side *PQ* is equal to the radius of the circle and the side *QR* is equal to its circumference (figure 1.1). He then considers squares inscribed within and circumscribed about the circle; these provide upper and lower bounds for the area of the circle, which are then systematically improved by doubling the number of sides in the inscribed and circumscribed figures. Thus, for example, the inscribed square *ABCD* can be replaced by an octagon constructed by bisecting $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, $\widehat{DA}$ and connecting the points *E, F, G,* and *H*. Similarly, a circumscribed square could be replaced by an octagon using the same sort of construction. By continuing the process of bisection, Archimedes constructs a sequence of approximating regular polygons such that the difference between the area of each polygon and the area of the circle is reduced by more than half with each successive term in the sequence. If the same procedure is followed for the circumscribing figures, two sequences will be generated that "compress" the area of the circle as they converge to a common limit. The proof then employs two reductio ad absurdum arguments showing that the area of the circle can be neither greater nor less than the area of the right triangle.[33]

It might initially seem that this result suffices to square the circle: after all, it gives a precise value for the circle's area, and although the right triangle used in the proof is not itself a square, it is a trivial matter to find a square equal in area to any triangle. However, the Archimedean theorem does not actually *construct* a triangle equal in area to the circle, but only shows that a right triangle with sides the length of the circle's radius and circumference is equal in area to the circle. To solve the problem of squaring the circle it would be necessary to devise a means of constructing "from scratch" a line equal in length to the circumference of a given circle, and it is this further task that cannot be solved within the framework of Euclidean geometry.

Consider, for example, the circle *BCD* with radius *AB*, as in figure 1.2. We achieve the quadrature of *BCD* only if we can construct a

33. The exhaustion generates an exact result by assuming (for purposes of reductio) that there is some finite difference, δ, between the circle and triangle. Then it can be shown that there is a polygon inscribed within or circumscribed about the circle that differs from the area of the circle by an amount less than δ. Either case leads to an absurdity, i.e., the circle being greater than a circumscribed polygon or less than an inscribed polygon. For a detailed discussion of the theorem and its proof, see Dijksterhuis [1956] 1987, chap. 6.

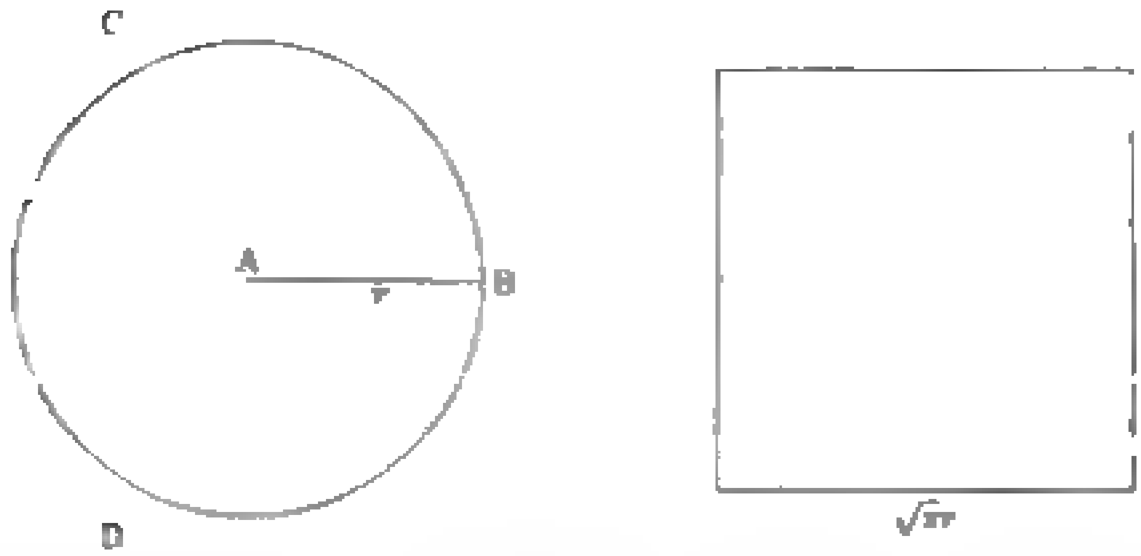

Figure 1.2

square with an area equal to it. Using the customary symbol π to designate the ratio of circumference to diameter in the circle and r for the magnitude of the radius AB, we know (on the basis of other results such as Archimedes' proposition above) that the area of the circle is πr^2. Thus, the square equal in area to the circle must have a side of length $\sqrt{\pi}r$. A line of length r is given in the statement of the problem, and it is a simple matter to construct a line equal to the square root of any given line,[34] so the quadrature of the circle is achieved if (and only if) we can construct a line of length π. It was not until the nineteenth century that the search for a construction of π was shown to be impossible,[35] and in Hobbes's day the problem still attracted the attention of leading mathematicians. The Dutch mathematician Willebrord Snell voiced the optimistic opinion that "such is the mechanical revolution of every circle, until it returns to the same point of the periphery, from whence the circumduction was begun; which indeed argues, and as it were sets before the eyes, that a right line may really be exhibited equal to the periphery" (Snell 1621, preface signature **2). His optimism was not shared by all writers on the subject. Wallis, for one, held that there could be no "geometrical" solution to the problem with classical

34. The extraction of the square root is essentially the problem of finding a mean proportional between two lines, one of which is taken as a unit. The construction in Euclid's *Elements* (6, prop. 13) suffices to extract the square root of any magnitude.

35. The impossibility of the construction of π was shown by the German mathematician C. L. F. Lindeman in 1882, with his proof that π is not an algebraic number, i.e., cannot be the root of an algebraic equation with integer coefficients. The difficulty is that the means of construction in classical geometry can only generate magnitudes corresponding to a subset of the algebraic numbers, but π is "transcendental"—which in effect means that it lies outside of a class of numbers that contains the classically constructable magnitudes as a subset. Thus, there is no means of constructing a line of length π.

methods, which is a view to which Descartes and Christiaan Huygens subscribed.[36]

It is important to stress here that the problem is unsolvable only with respect to a set of principles governing the construction of geometric magnitudes. The framework of Euclidean geometry permits only "compass and rule" constructions, which is to say constructions involving only the description of circles and the drawing of straight lines. The familiar compass and straight rule are the only geometric implements permitted in the Euclidean solution of problems, and the impossibility of squaring the circle is to be understood as the inability of compass-and-rule constructions to deliver a true quadrature. The Greeks were familiar with more powerful problem-solving methods, and they distinguished between "plane" solutions, which relied only upon compass-and-rule constructions, and "solid" solutions, which used curves (such as conic sections) generated by the intersection of planes and solids, and "linear" solutions, which employed curves (such as the spiral) whose definition is still more complex.[37]

Methods that extend beyond the austere Euclidean means can, in fact, square the circle, and it is instructive to consider one such method based on a special curve known as the "quadratrix of Hippias." It is unclear when the curve was discovered, or exactly who Hippias was (Knorr 1986, 80–82). No original works of Hippias survive and we know of the curve and its applications only through the report of later mathematical writers such as Pappus of Alexandria, who provides the best account of this curve in his *Mathematical Collection*. Let the

36. Wallis's doubts about the possibility of finding an exact value for π are voiced in chapter 24 of *MU* (*OM* 1:132). Mancosu (1996, 77–79) has shown that Descartes's pessimism is rooted in his mathematical epistemology. Huygens's opinion is explored in Breger 1986, with reference to the work of Leibniz and James Gregory.

37. Pappus reports in book 3 of his *Mathematical Collection* that "[t]he ancients maintained that there are three types of problems, one of which is called plane, another solid, and the third linear. Therefore those problems which can be solved by right lines and the circumferences of circles are justly called plane, since the lines by which they are solved have their origin in the plane. But those problems whose resolution is found by one or more sections of the cone are called solid problems, for it is necessary in the construction to use the surfaces of solids, namely cones. There remains the third type of problem, which is called linear; for in these cases the construction requires curves beyond those already described, which curves have a more varied and forced origin. Among these curves are the spirals, quadratrices, conchoids, and cissoids, all of which have many wonderful properties" (Pappus of Alexandria [1875] 1965, 1:55). For more on the classification of problems see Heath [1921] 1981, chap. 7. This issue will be of interest in chapter 5 when we investigate Hobbes's views on his "method of motions" and its comparison with the techniques of analytic geometry.

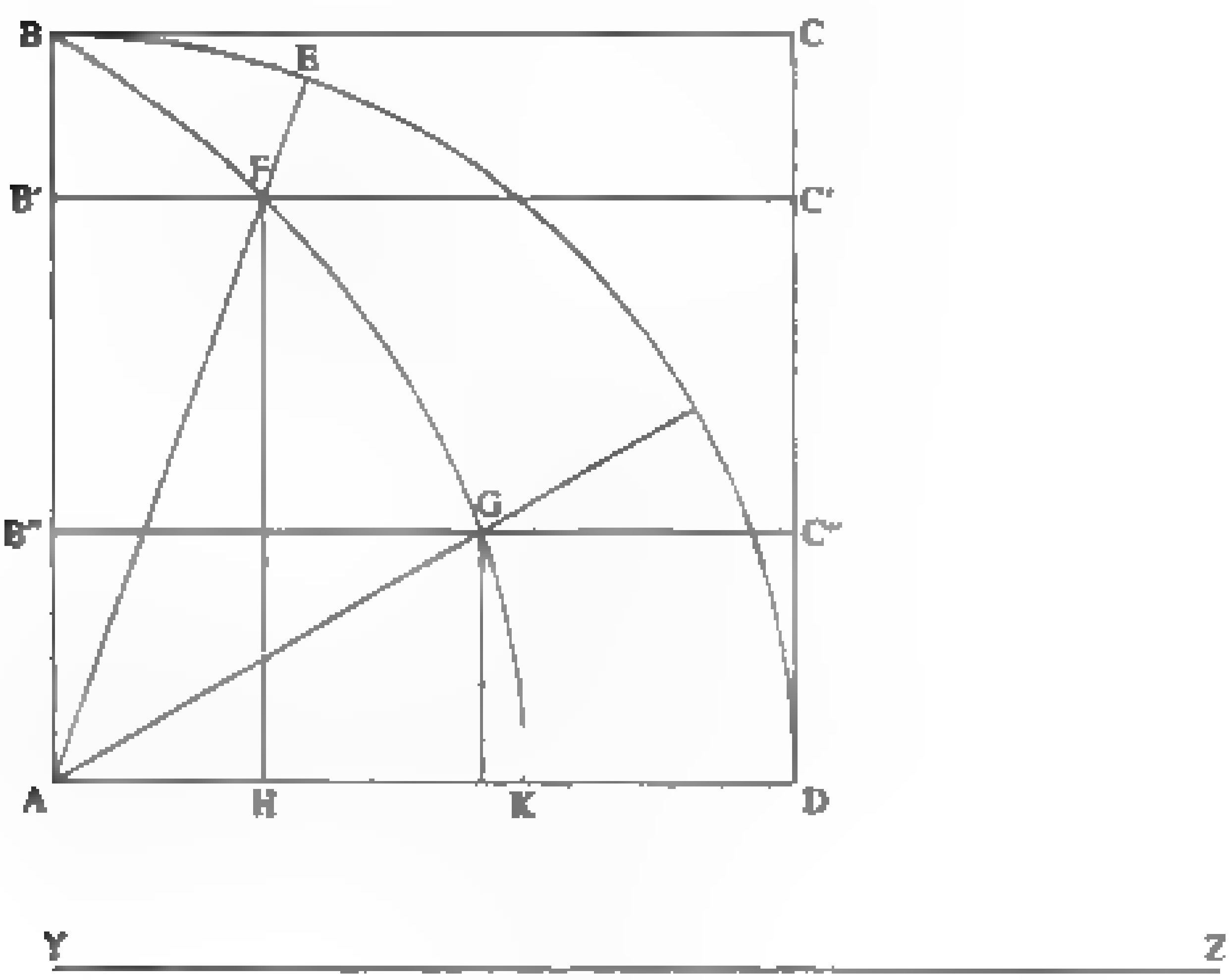

Figure 1.3

square *ABCD* be given as in figure 1.3, and describe within it the circle quadrant *BED* with center *A*. Suppose that the radius *AB* describes $\widehat{BED}$ with uniform motion, and that at the same time the line BC moves uniformly and always parallel with itself into the position *AD*. Let *B'C'* and *B"C"* be two positions of the line *BC* as it transverses the distance *BA*. Throughout the time during which these lines move, they will intersect, as for example at the points *F* and *G*. The collection of points of intersection (called the locus of intersection) defines the curve known as the quadratrix. By definition, the curve has the property that $\angle BAD{:}\angle EAD = \widehat{BED}{:}\widehat{ED} = AB{:}FH$. Let *K* be the point of intersection of the quadratrix and the radius *AD*. Then it can be proved (by a complex exhaustion proof, whose details I omit) that $\widehat{BED}{:}AB = AB{:}AK$, so that the radius *AB* is a mean proportional between $\widehat{BED}$ and the line *AK*.[38] The problem of quadrature is then easily solved. Because $AK{:}AB = AB{:}\widehat{BED}$, we can use the lines *AK* and *AB* to construct a straight line *YZ* equal to the arc of the quadrant; that is to say, $AK{:}AB :: AB{:}YZ$, where *YZ* is equal in length to $\widehat{BED}$. The line *YZ* is one-fourth the circumference of the circle, so that taking a line four times the length of *YZ* will give a straight line equal to the circumference; but this new line and the radius can be used to construct a right

38. For the details and a discussion of the quadratrix see Knorr 1986, 82–83 and chap. 6, and Heath [1921] 1981, 1:227–28. The construction of the quadratrix and proofs of its properties can also be found in Pappus [1875] 1965, 1:252–58.

triangle equal in area to the circle, by proposition 1 of Archimedes' *On the Measurement of the Circle*.

The quadratrix is not a properly geometric curve in the strictest classical sense because it cannot be constructed with compass and rule. The definition of the curve employs the consideration of two uniform motions—one circular and the other rectilinear—and it further requires that the ratio of equality between a circular arc and a straight line be unproblematically assigned throughout the generation of the curve. Indeed, the question of whether lengths of circular arcs can be exactly compared with straight lines is the very question at issue in the quadrature of the circle. It is possible to construct any number of points on the curve by finding the intersection of circular radii with lines parallel to *BC*, but such "pointwise" construction falls short of a general construction of the curve.[39]

1.2.2.2 THE TRISECTION OF THE ANGLE The problem of trisecting an arbitrary angle presumably has its roots in the project of constructing regular polygons and solids.[40] Euclidean means permit the construction of regular polygons of 2^n sides, for n integer > 1. In addition, planar methods permit the construction of the equilateral triangle and regular pentagon. These means are exploited in the *Elements* to construct the five "Platonic solids" (tetrahedron, cube, octahedron, dodecahedron, and icosahedron), and the question naturally arises whether other regular figures and solids may be constructed. The

39. Clavius, interestingly enough, admits the pointwise construction as fully geometrical while rejecting the definition in terms of motions. He considers the case where the side *AB* is divided into eight equal segments and the curve *BK* approximated (I have altered his labeling of the lines in his diagram to conform to those I am using). "But because these two uniform motions (one of which is through the circumference *[BD]* and the other through the right lines *[BC]* and *[CD]*) cannot be effected unless the proportion between the circular arc and the right line is known, this construction is justly reprehended by Pappus, since when the proportion is unknown, so also is that which is to be investigated by this line. Thus we will describe the same quadratrix geometrically in this way. Let the arc *[BD]* be divided in any number of equal parts, and let both sides *[AB]*, *[BC]* be divided in as many equal parts. The easiest division will be if the arc *[BD]*, and both sides *[AB]*, *[BC]* are first divided in half, and then both of these halves are again bisected, and each of these parts is again bisected, and this process is continued as far as one might wish. And the more divisions are effected, the more accurately the line will describe the quadratrix. To avoid confusion, we have divided both the arc *[DB]* and the two sides *[AD]*, *[BC]* in eight equal parts" (1612 1:296). See Mancosu 1996, 74–77, for an account of Clavius's approximation procedure, which is very similar to the νεῦσις constructions taken up in section 1.2.2.2 of this chapter.

40. Regular polygons are those that have equal sides and angles; a regular solid is one whose faces are equal regular polygons.

division of angles is an essential part of the project of constructing regular figures, and if it were possible to divide the circle into any number of equal arcs, then by joining the successive points of division by chords one could produce any desired regular polygon.

The bisection of any angle is an elementary problem (*Elements* 1, prop. 9), and continued application of this result allows the construction of any regular polygon with 2^n sides. But the problem of angular trisection is more difficult. As Proclus notes in commenting on the Euclidean problem of bisecting any given angle:

> To divide in any ratio that might be chosen—as into three, or four, or five equal parts—goes beyond the present means of construction. We can divide a right angle into three parts by using some of the theorems that follow, but we cannot thus divide any acute angle without resorting to other lines that are mixed in kind. This is shown by those who have applied themselves to the problem of trisecting a given rectilinear angle. Nicomedes made use of conchoids—a form of line whose construction, kinds, and properties he has taught us, being himself the discoverer of their peculiarities—and thus succeeded in trisecting the angle generally. Others have done the same thing by means of the quadratrix of Hippias and that of Nicomedes, they too using mixed lines, namely, the quadratrices. Still others have started from the spirals of Archimedes and divided a given rectilinear angle in a given ratio. The thoughts of these men are difficult for a beginner to follow, and so we pass them by here. (Proclus 1970, 211–12)

As Proclus indicates, the quadratrix also permits the trisection of any angle, and indeed it allows an arbitrary angle to be divided into any number of equal parts. For example, suppose that $\angle EAD$ in figure 1.4 is any acute angle and let FH be divided at F' so that $FF':F'H$ is the desired ratio.[41] Draw $B''C''$ parallel to AD and through F' to cut the curve at L, and join AL to meet the circle at N. Then $\angle EAN:\angle NAD$ will be in the ratio $FF':F'H$ because $\angle EAN:\angle NAD = \widehat{EN}:\widehat{ND} = FF':F'H$.

Another approach to the problem of angle trisection uses a technique called "reduction to a νεῦσις." The Greek term "νεῦσις" means "verging" or "approach," and a νεῦσις problem is one in which a line

41. The restriction of $\angle EAD$ to an acute angle imposes no loss of generality on the theorem: a right angle can be trisected, and any obtuse angle can be divided into a right angle and an acute angle.

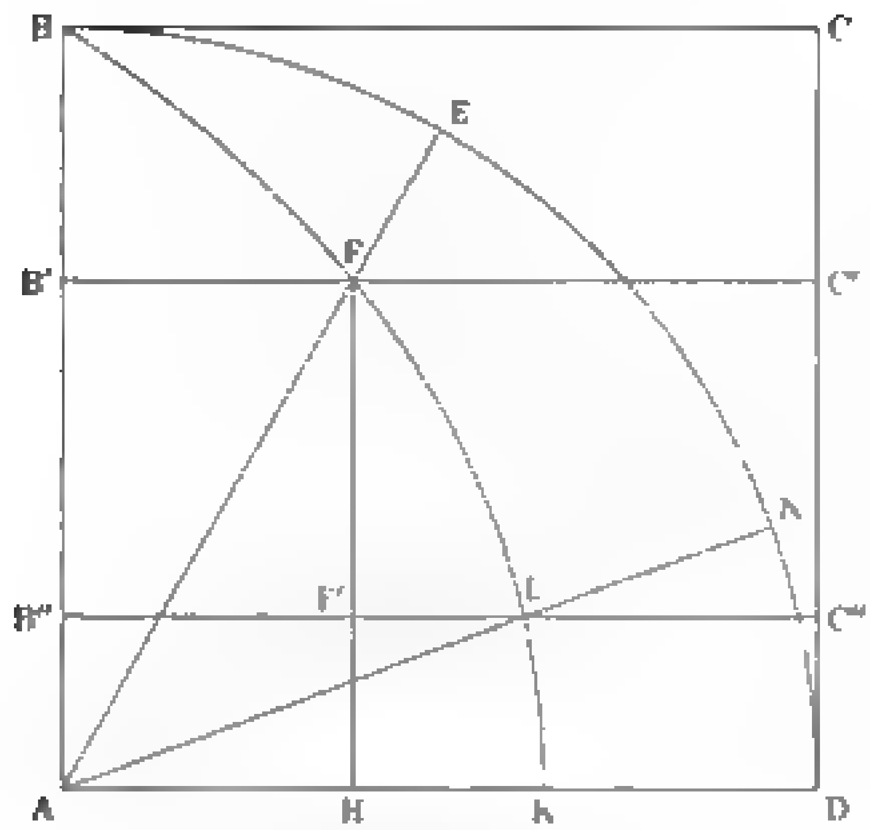

Figure 1.4

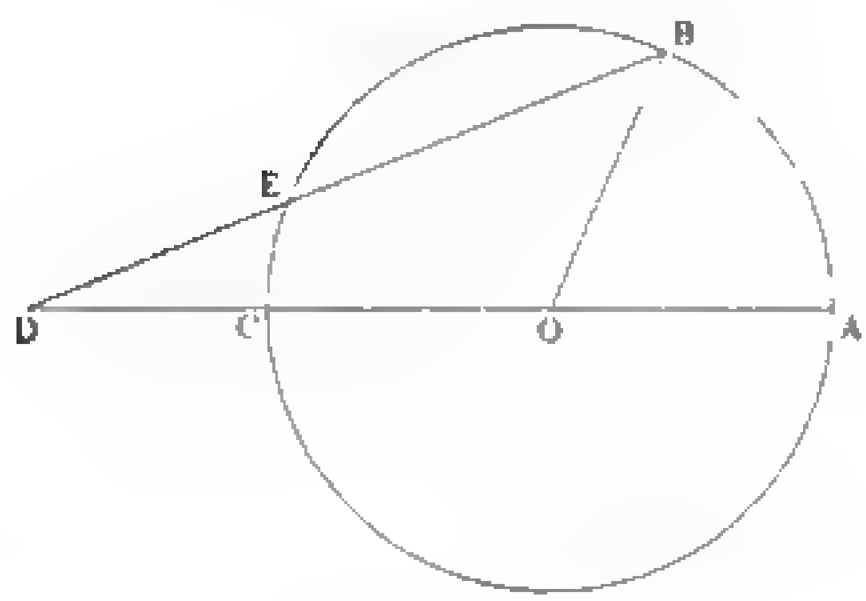

Figure 1.5

is inserted into a construction in such a way that it verges or inclines toward a point that has a desired property. For example, take the angle *BOA* in figure 1.5. We trisect it by first constructing the circle *ABC* with center *O*, then draw a line from *B* to intersect the diameter *OC* extended in the point *D* such that *DE* is equal to the radius of the circle. Then $\angle EDC$ will be one-third of $\angle BOA$.[42] The difficulty in the construction is finding the point *D*, which cannot be constructed with the means available to classical geometry. By trial and error, points on

42. That $\angle EDC$ is one-third of $\angle BOA$ follows from the properties of isosceles triangles. If we connect *OE* to form the isosceles triangles *OED* and *OEB*, we have $\angle COE = \angle EDC$, and $\angle BEO = \angle EBO = 2(\angle EDC)$; Similarly, $\angle BOA = \angle EBO + \angle EDC = 3(\angle EDC)$ and $\angle EDC$ trisects $\angle BOA$. The theorem needed for this proof is *Elements* (1, prop. 32): "In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles."

the circle can be investigated to see whether they satisfy the condition of the construction, and with sufficient effort the location of D can be pinned down within any desired degree of precision (hence the term "νεῦσις"), but the actual construction of a line through D cannot be effected by planar means or conic sections.[43]

1.2.2.3 THE DUPLICATION OF THE CUBE The third and final great problem from antiquity is that of constructing a cube double in volume to a given cube.[44] This is often known as the "Delian problem," because of the legend that the problem arose when the oracle at Delos commanded that the cubiform altar in the temple was to be doubled. Theon of Smyrna, a second-century neo-Platonist, records the legend in his handbook of mathematical material relating to the study of Plato. He reports that:

> when the god pronounced to the Delians in the matter of deliverance from a plague that they construct an altar double of the one that existed, much bewilderment fell upon the builders who sought how one was to make a solid double of a solid. Then there arrived men to inquire of this from Plato. But he said to them that not for want of a double altar did the god prophesy this to the Delians, but to accuse and reproach the Greeks for neglecting mathematics and making little of geometry. (Knorr 1986, 21)

The problem reduces to the finding two mean proportionals between a line and its double. Let the side of the given cube be S_1; then the problem is to find a line S_2 such that $(S_2)^3 = 2(S_1)^3$. If we construct two lines X and Y such that $S_1{:}X = X{:}Y = Y{:}2(S_1)$, then by compounding ratios we have $(S_1{:}X)^3 = (S_1{:}X)\ (X{:}Y)\ (Y{:}2(S_1))$, so that $(S_1)^3{:}X^3 = S_1{:}2(S_1)$ and thus $X^3 = 2(S_1)^3$ so that X is the line S_2 that we seek.[45] The unsolvability of the Delian problem (relative to Euclidean means) derives from the same kind of difficulty we encountered earlier in the case of the quadrature of the circle. As before, there are more complex curves that can permit a solution, but the use of "planar" means does not permit the problem to be solved. The class of magnitudes constructible from Euclidean operations includes only those that can be

43. For an interesting elementary proof of this result, see Quine 1990.

44. See Knorr 1986, 17–25, and Saito 1995 for more detailed accounts of this problem.

45. In fact, the problem easily generalizes to cover cases of lines in any ratio, rather than just the ratio of 1:2. In the general case, the cubes of the lines S_1 and S_2 may be in any ratio whatever.

obtained by a finite application of the operations of addition, subtraction, multiplication, division, and the extraction of square roots. But the magnitudes required to solve the Delian problem cannot be so obtained.

This concludes our account of the great problems to which Hobbes claimed solution, but we must now consider the more important mathematical developments of his own era. As we will see, the attempt to find solutions to these classical problems led to an important expansion of mathematical methods, and it is essential that we understand the relationship between these new methods and Hobbes's conception of mathematics.

1.2.3 *The Seventeenth-Century Mathematical Background*

Hobbes's life spanned an enormous transformation in European mathematics. When he was born in 1588 the study of mathematics centered on the classical works of Greek antiquity and most of the mathematical work of the period was devoted to producing editions and commentaries on classical authors. By the time of Hobbes's death in 1679 the mathematical landscape had changed dramatically: Isaac Newton had long since circulated his first papers on the method of fluxions, G. W. Leibniz had applied himself to mathematics, and the great advancement now known as the calculus was well under way. The two most significant mathematical changes during this period were the development of analytic geometry and the rise of infinitesimal techniques, particularly the method of indivisibles. Indeed, it is no exaggeration to say that modern mathematics begins with the advent of these two methods. Hobbes's own conception of mathematics cannot be understood without reference to these methods, and my aim in this section is to outline the fundamental ideas behind them while indicating the controversies they provoked. In general, these controversies were the product of two conflicting motivations shared by many in the mathematical community: the desire to pursue new results (and particularly the solution of the outstanding classical problems) and the desire to maintain the standards of rigorous demonstration upheld by Greek geometry of the classical period.

1.2.3.1 THE ANALYTIC ART AND THE STATUS OF GEOMETRY The publication of Descartes's *Géométrie* in 1637 is usually regarded as the advent of analytic geometry, and is generally hailed as a significant mathematical advance. Hobbes, however, disparaged ana-

lytic geometry and saw it as something akin to a mathematical perversion: in his estimation, the "modern analytics" were a corruption that impeded the progress of geometry. Hobbes's grounds for this judgment will be examined in chapter 5, but in order to clarify matters we must first investigate some of the history of analytic geometry. The fundamentals of the analytic approach are familiar and need no detailed account here.[46] The essential point is that algebraic operations (addition, subtraction, multiplication, division, and the extraction of roots) are interpreted as geometric constructions and curves are then identified with indeterminate equations in two unknowns. Then, algebraic techniques for solving equations are applied to the investigation of geometric curves, and the nature of a curve can be systematically explored by examining the structure of the equation with which the curve is identified. François Viéte's *Isagoge in Artem Analyticem* (Viéte 1646) is a key text in the development of analytic geometry, most notable for its expansion of algebraic techniques and their application to geometric problems.[47] In Britain, William Oughtred's 1631 treatise *Clavis Mathematicae* introduced a generation of mathematicians to algebraic methods and their use in geometry, while the unpublished letters and manuscripts of Thomas Harriot furthered the progress of the subject. It is difficult to discern the extent to which Descartes is indebted to Viéte, Oughtred, or anyone else, but there is no question that all three authors promoted the incorporation of algebraic techniques into geometry.[48] This fusion of algebraic and geometric methods allows

46. See Boyer 1956 for the standard account of the development of the subject, as well as Mahoney 1994, chap. 3. Historians of mathematics also credit Fermat with the invention of analytic geometry, but his work is not prominent in Hobbes's polemics against analytic methods. I will therefore ignore Fermat's contributions in the course of this study. The most accessible recent study of Descartes's geometry is Mancosu 1992, which can be supplemented by Mancosu 1996, chap. 3.

47. Indeed, Viéte's work is so fundamental to the development of modern algebra that Jacob Klein called him "the true founder of modern mathematics" (Klein 1968, 5). For more on seventeenth-century algebra and its philosophical interpretation, see Mahoney 1980 and Krämer 1991, 124–51.

48. Descartes is famously reticent about sources for his mathematical work and never acknowledged a significant debt to other mathematicians, as improbable as this may seem. For purposes of our investigation, the best statement of Descartes's attitude comes from a letter of Pell to Cavendish dated 12/22 March 1646 and reporting a conversation with Descartes on mathematical matters. Pell writes: "I perceive he demonstrates not willingly. He sayes he hath penned very few demonstrations in his life (understand after ye style of ye old Grecians which he affects not) THAT he never had an Euclide of his owne but in 4 dayes, 30 yeares agoe. . . . Of all ye Ancients he magnifies none but Archimedes, who he sayes, in his bookes de Sphaera & Cylindro and a piece or two more, shows himselfe fuisse bonum Algebraicum & habuisse vere-magnum ingenium. I

systematic and relatively simple solutions to problems that require elaborate constructions with compass and rule in classical geometry. Moreover, analytic geometry can classify curves by their characteristic equations and study curves that are more complex than those accessible to classical investigation. The role of algebraic analysis in the new geometry is summed up in Descartes's remark in book 2 of the *Géométrie:*

> I could give here many other means for tracing and conceiving curves, which means would become more and more complex by degrees to infinity. But to bring together all those that are in nature and to distinguish them in order into certain genres, I know of no better way than to say that all those points of curves we can call geometric, that is to say that have a precise and exact measure, must necessarily have a relation to all the points of a right line, which can be expressed by some single equation. (*AT* 6:392)

The Cartesian program for geometry classifies as properly geometric (as opposed to "mechanical") any curves that have a "precise and exact" measure. This rather vague criterion is elucidated slightly by Descartes's declaration that geometric curves are those that can be described by a regular motion or series of motions (*AT* 6:390). Descartes implicitly assumes that any curve representable algebraically could be traced by such continuous motions and therefore admitted as geometric, and conversely that any geometric curve could be represented by an equation.[49] In the solution of problems, Descartes constrains the choice of curves by the requirement that the simplest curve (i.e., the curve whose equation is of lowest degree) be employed. This restriction has a clear antecedent in Pappus's distinction between planar, solid,

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will not trouble you of what he said of Vieta, Fermat and Roberval and Golius: Of Mr. Hobbes I durst make no mention to him." British Library MS. Add. 4280, f. 117ʳ. The letter is partially reprinted in *AT* 4:729–32 and fully reprinted in Hervey 1952, 77–79.

49. See Bos 1981 for a discussion of Descartes's program for geometry and the difficulties surrounding his classification and representation of curves. As Bos notes, Descartes "could not simply take as 'geometric' all curves that admit an algebraic equation; if he were to adopt this criterion, Descartes could no longer claim that he was doing geometry" (Bos 1981, 305). The result is that the use of a purely algebraic criterion for geometric curves is merely implicit in Descartes's work. Mancosu 1996, chap. 3, contains an insightful study of this topic with respect to Descartes's mathematical epistemology.

and linear problems, although it is now formulated in terms of the relative complexity of equations.

Another important difference between analytic and classical methods concerns the manner in which algebraic operations are interpreted in geometry. Classically, the geometric multiplication of two lines yields a rectangle, or the product of three lines a solid. But Descartes interprets multiplication as an operation that leaves the dimension of the product homogeneous with that of the multiplicands. Just as the product of two numbers is a number, Cartesian analytic geometry treats the product of two lines as a line. And in general, all operations in analytic geometry are operations on line segments that result in new line segments. This is the import of Descartes's declaration at the beginning of the first book of the *Géométrie:*

> All problems of geometry can easily be reduced to such terms that there is need only to know the length of certain right lines in order to construct them. Just as all of arithmetic is composed of only four or five operations, which are addition, subtraction, multiplication, division, and the extraction of roots (which we can take for a sort of division), so also in geometry we need do nothing more to find the sought lines than to add or subtract other lines; or else, having one line that I shall call unity (in order to relate it as closely as possible to numbers), and which one can ordinarily be taken arbitrarily, and having two other lines given, to find a fourth line that is to one of the given lines as the other stands to unity, which is the same as multiplication; or again to find a fourth line which is to one of the given lines as unity is to the other, which is the same as division; or finally to find one, two, or more mean proportionals between unity and some other line, which is the same as extracting the square root, or cube root, etc. And I will not hesitate to introduce these terms from arithmetic into geometry, in order to make matters more intelligible. (*AT* 6:370)

The approach at work here exploits an algebraic treatment of line segments to allow the product of two magnitudes to be compared with either of its multiplicands. Given an arbitrarily assigned unit segment, the proportional relation $1:a :: b:ab$ makes all four quantities in the proportion mutually comparable as in figure 1.6. Moreover, the quantity ab can be interpreted as a line segment constructed from two similar triangles with corresponding sides $1, b$ and a, ab respectively.

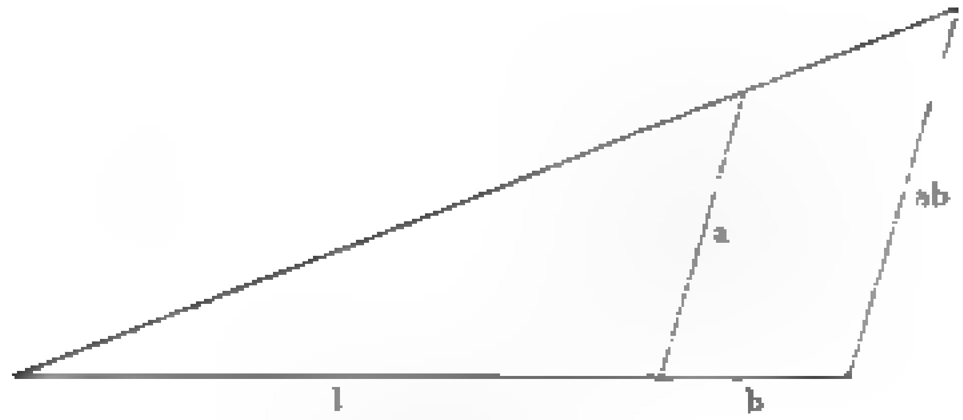

Figure 1.6

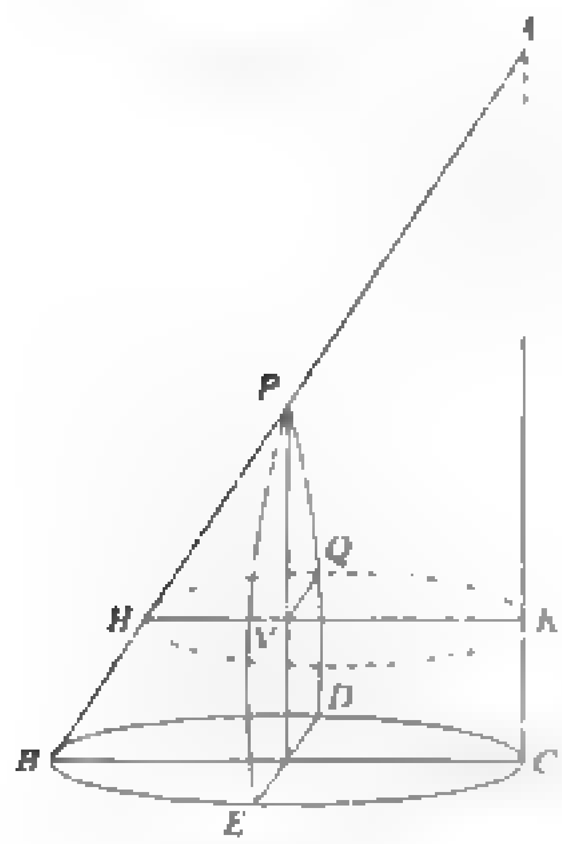

Figure 1.7

This conception of geometry is underwritten by a strong thesis on the unity of arithmetical and geometric magnitudes. Descartes sees nothing peculiarly arithmetical about the operation of addition, or anything uniquely geometrical about the extraction of roots. The resulting application of algebra to geometry therefore treats algebra as a science of magnitude in general, and the specifically geometric content of a problem is removed (and, at least in Descartes's view, the problem is rendered more intelligible) when it is represented as a relation among various abstract magnitudes.

The project of studying geometric objects by inquiring into the relationship between right lines obviously did not originate with Descartes. The classic theory of conic sections, for example, begins with the oblique cone *ABC*, with vertex *A* and circular base *BC* (figure 1.7). Through any point *P* on the element *AB*, we pass a plane cutting the base in the chord *ED*. The section *DPE* of the cone is a parabola, which has the property that products of the lines *QV, PV, PA, BC, AC,* and *BA* all stand in a specific ratio. In this case, taking any point *Q*

on the parabola, we have the circle *HKQ* with diameter *HK* and perpendiculars *VQ*, *PV*, such that $QV^2 : PV = (PA \times BC^2):(AC \times BA)$. Where Descartes differs from such a traditional understanding is first in presenting all curves in a simple two-dimensional plane rather than as constructions from solids, which thereby allows the curve to be represented as the solution to an algebraic equation in two unknowns forming the axes of the coordinate system. He also introduces the assumption of the homogeneity of the magnitudes under study, so that lines can be multiplied together any number of times to form curves of fourth and higher dimension.

The advent of analytic methods provoked a philosophical debate on the question whether arithmetic or geometry was the genuinely foundational discipline in mathematics.[50] Classical mathematicians distinguished discrete quantity ("number") from continuous quantity ("magnitude"), declaring the former to be the object of arithmetic and the latter to be the proper object of geometry. Classically, then, geometry and arithmetic are distinct sciences with no common object, so there is no need to ask which is the more fundamental science. This situation changed with the development of analytic geometry. Many interpreted algebra as a kind of generalization of arithmetic, and it was often characterized as the "arithmetic of species," in which variables such as x or ξ were taken as general representatives of kinds or species of quantities.[51] In this scheme, the basic principles of algebra were seen as deriving from arithmetic, and the prominence of algebraic methods in analytic geometry led some to conclude that geometry must, in some important sense, be based on arithmetic.

Wallis argued for the primacy of arithmetic over geometry in his 1657 *Mathesis Universalis*. This work (which began as Savilian lectures) marshals philosophical, historical, and philological arguments to show that all of mathematics is ultimately founded upon arithmetic. Indeed, Wallis's point of view is evident in the full title of the *Mathesis Universalis*, which promises (among other things) "a complete arithmetical work, presented both philologically and mathematically, encompassing both the numerical and the specious or symbolic arithme-

50. Pycior 1987 and Sasaki 1985 are valuable studies of this debate, which can be supplemented by the detailed study of Barrow and Wallis in Maierù 1994 and the account of the battle between ancients and moderns in Hill 1997.

51. Viète, for example, declares that his analytic method "no longer limits its reasoning to numbers, a shortcoming of the old analysts, but works with a newly discovered logic of species, which is far more fruitful and powerful than numerical logistic for comparing magnitudes with one another" (Viete 1646, 1).

tic, or geometric calculus."[52] Part of this program to elevate arithmetic to the status of "universal mathematics" involves arguing that geometric results can be achieved more perspicuously and naturally by the use of arithmetical arguments. Thus, for example, Wallis devotes the twenty-third chapter to a series of "arithmetical" demonstrations of results from the second book of Euclid's *Elements,* which he takes to illustrate his contention that the important results in geometry are ultimately founded upon arithmetical principles. In his philosophical case for the primacy of arithmetic, Wallis admits that such geometric terms as *root, square,* and *cube* appear in algebra, but denies that this should lead to the conclusion that algebra is based on geometry. Although some authors have drawn this conclusion, Wallis claims that geometry ultimately takes its principles from arithmetic because the principles of arithmetic are presupposed in any geometrical demonstration.[53] Part of his reasoning on this point is the argument that universal algebra is fundamentally arithmetical and not geometrical. He insists:

> Indeed many geometric things can be discovered or elucidated by algebraic principles, and yet it does not follow that algebra is geometrical, or even that it is based on geometric principles (as some would seem to think). This close affinity of arithmetic and geometry comes about, rather, because geometry is as it were subordinate to arithmetic, and applies universal principles of arithmetic to its special objects. For, if someone asserts that a line of three feet added to a line of two feet makes a line five feet long, he asserts this because the numbers two and three added together make five; yet this calculation is not therefore geometrical, but clearly arithmetical, although it is used in geometric measurement. For the assertion of the equality of the number five with the numbers two and three taken together is a general assertion, applicable to other kinds of things whatever, no less than to geo-

52. The full title of the work in Wallis's *Opera Mathematica* of 1693 is *Mathesis Universalis: Sive, Arithmeticum Opus Integrum, Tum Philologice, tum Mathematice traditum, Arithmeticam tum Numerosam, tum Speciosam sive Symbolicam complectens, sive Calculum Geometricum; tum etiam Rationum Proportionumve traditionem; Logarithmorum item Doctrinam; aliaque, quae Capitum Syllabus indicabit* (*OM* 1:11).

53. "Because some take the geometric elements for the basis of all of mathematics, they even think that all of arithmetic is to be reduced to geometry, and that there is no better way to show the truth of arithmetical theorems than by proving them from geometry. But in fact arithmetical truths are of a higher and more abstract nature than those of geometry. For example, it is not because a *two foot line added to a two foot line makes a four foot line* that *two and two are four,* but rather because the latter is true, the former follows" (*MU* 11; *OM* 1:53).

> metrical objects. For also two angels and three angels make five angels. And the very same reasoning holds of all arithmetical and especially algebraic operations, which proceed from principles more general than those in geometry, which are restricted to measure. (*MU* 11; *OM* 1:56)

Isaac Barrow, the first Lucasian Professor of Mathematics at Cambridge, rejected this reasoning. In fact, he responded directly to Wallis's argument and attempted to show that geometric principles must be presupposed as the foundation of all mathematics. In the third of his *Lectiones Mathematicae* Barrow considers Wallis's argument for the priority of arithmetic and issues the following rebuttal:

> To this I respond by asking, How does it happen that a line of two feet added to a line of two palms does not make a line of four feet, four palms, or four of any denomination, if it is abstractly, i.e., universally and absolutely true that two plus two makes four? You will say, This is because the numbers are not applied to the same matter or measure. And I would say the same thing, from which I conclude that it is not from the abstract ratio of numbers that two and two make four, but from the condition of the matter to which they are applied. This is because any magnitude denominated by the name *two* added to a magnitude denominated *two* of the same kind will make a magnitude whose denomination will be *four*. Nor indeed can anything more absurd be imagined than to affirm that the proportions of magnitudes to one another depend upon the relations of the numbers by which they may be expressed. (*LM* 3, 53)

Barrow's case for the primacy of geometry hinges on the claim that numbers, far from being self-subsistent objects, are mere symbols whose content derives from their application to continuous geometric magnitude. To put it another way, there are no "numbers in the abstract" to serve as the object of arithmetic, except those that arise from the consideration of homogeneous magnitude and its division.[54] Bar-

54. "I say that mathematical number is not something having existence proper to itself, and really distinct from the magnitude it denominates, but is only a kind of note or sign of magnitude considered in a certain manner; so far as the magnitude is considered as simply incomposite, or as composed out of certain homogeneous equal parts, every one of which is taken simply and denominated a unit. . . . For in order to expound and declare our conception of a magnitude, we designate it by the name or character of a certain number, which consequently is nothing other than the note or symbol of such magnitude so taken. This is the general nature, meaning, and account of a mathematical

row took this case to the extreme of denying that algebra is a mathematical science at all, classifying it as simply a part of logic or a set of rules for manipulating symbols. He distinguishes two branches of algebra, analytics and logistics. The former, he says "seems to be no more proper to mathematics than to physics, ethics, or any other science. For it is only a part or species of logic." The latter is no part of mathematics "because it has no object distinct and proper to itself, but only presents a kind of artifice, founded on geometry (or arithmetic), in which magnitudes and numbers are designated by certain notes or symbols, and in which their sums and differences are collected and compared" (*LM* 2, 46). The difference of opinion between Barrow and Wallis is significant for understanding Hobbes's relationship to seventeenth-century mathematics. Hobbes developed views on this particular question that are close to Barrow's, and his account of the nature of mathematics becomes more intelligible if we see him as responding to the concerns that produced this dispute over the relative priority of arithmetic and geometry. This will emerge more clearly in chapter 3, when we investigate Hobbes's philosophy of mathematics.

1.2.3.2 THE METHOD OF INDIVISIBLES The development of the method of indivisibles was another pivotal advance in seventeenth-century mathematics, notable both for providing a wealth of new results and its share of controversy. The first exposition of the new method was in Bonaventura Cavalieri's *Geometria indivisibilibus continuorum nova quadam ratione promota* (Cavalieri 1635). The method plays upon the intuition that we can reason about the area of a figure by considering the lines it contains, which Cavalieri calls the indivisibles of the figure.[55] Cavalieri was cautious about claiming that these indivisibles actually compose the figure, although he did seek analogies between the composition of cloth out of threads and the relationship of a figure to its indivisibles.[56] Instead of simply taking indivis-

number" (*LM* 3, 56). See Mahoney 1990, 186–89, for more on Barrow's account of number.

55. In a perfectly analogous manner the motion of a plane through a solid could produce all the planes of the solid, or the indivisibles of the solid. For more on Cavalieri and his method see Andersen 1985, De Gandt 1991, Festa 1992, and Giusti 1980.

56. In his *Exercitationes Geometricae Sex*, he declares that "it is manifest that we can conceive of plane figures in the form of cloth woven out of parallel threads, and solids in the form of books, which are built up out of parallel pages." He nevertheless quickly adds: "But the threads in a cloth and the pages in a book are always finite and have some thickness, while in this method an indefinite number of lines in plane figures (or planes in solids) are to be supposed, without any thickness" (Cavalieri 1647, 3–4).

Figure 1.8

ibles as infinitely small components of finite magnitudes, Cavalieri sought to introduce indivisibles as a new species of magnitude that could be brought within the purview of the classical theory of ratios.

In Cavalieri's terminology, "all the lines" of the plane figures *ABCD* and *EFGH* in figure 1.8 are produced by the transit of the line *LM* (called the *regula*) through the figures. Significantly, Cavalieri avoids the question of whether there are an infinite number of indivisibles produced by the transit of the regula *LM* or whether these indivisibles are infinitely small when compared with the figures, apparently hoping that his method would be acceptable on any resolution of the problems surrounding the infinite. He speaks vaguely of an "indefinite" number of lines contained within a figure, and stresses that the ratios can be compared either "collectively" (as one collection of indivisibles to another), or "distributively" by comparing corresponding lines singly.[57] His appeal to continuous motion arises from similar concerns: he seems to have regarded this as a relatively unproblematic concept that can sidestep thorny questions concerning infinity. After all, anyone will admit that a line can pass through the figure, and the intersection of the line and figure will produce, in some fairly innocuous sense, "all the lines" contained in the figure.

Given this starting point Cavalieri treated "all the lines" of the figure as a new species of geometric magnitudes that could be dealt with according to the theory of magnitudes in book 5 of Euclid's *Elements*. His strategy is to establish a ratio between the indivisibles of two fig-

57. These two different presentations of the method appear more clearly in his 1647 *Exercitationes Geometricae Sex*, although the second is also contained in the sixth and last book of the *Geometria*. He explains the distinction between the two procedures thus: "The first method proceeds by the first kind of reasoning, and compares aggregates of all the lines of a figure or all the planes of a solid to one another, however many they may be. But the second method uses the second kind of reasoning, and compares single lines to single lines and single planes to single planes, lying in the same direction" (Cavalieri 1647, 4).

ures (either distributively or collectively), and then to conclude that the same ratio holds between the areas of the figures. This is the import of the "very general rule" he announces in the first of his *Exercitationes Geometricae Sex:*

> From these two [ways of comparing indivisibles] a single and most general rule can be fashioned, which will be a summary of all this new geometry, namely this: *Figures, both plane and solid, are in the same ratio as that of their indivisibles compared with one another collectively or . . . distributively.* (Cavalieri 1647, 6–7)

Cavalieri's evident caution on foundational matters was not shared by other mathematicians of the seventeenth century, most notably Wallis.[58] In Wallis's treatment, geometric problems are represented analytically and solved by determining the relationship between the infinite sums of infinitely small indivisibles that compose the figures. As an example consider his approach to the quadrature of the cubic parabola in his *Arithmetica Infinitorum.* He begins with arithmetical results in proposition 39, observing that:

$$\frac{0+1}{1+1} = \frac{1}{2} = \frac{1}{4} + \frac{1}{4}$$

$$\frac{0+1+8}{8+8+8} = \frac{9}{24} = \frac{3}{8} = \frac{1}{4} + \frac{1}{8}$$

$$\frac{0+1+8+27}{27+27+27+27} = \frac{36}{108} = \frac{4}{12} = \frac{1}{4} + \frac{1}{12}$$

$$\frac{0+1+8+27+64}{64+64+64+64+64} = \frac{100}{320} = \frac{5}{16} = \frac{1}{4} + \frac{1}{16}$$

From these initial cases, Wallis concludes "by induction" that as the number of terms in the sums increases, the ratio approaches arbitrarily near to the ratio 1:4. Proposition 41, which he takes to follow obviously from proposition 39, asserts that:

58. Andersen 1985, sec. 10; De Gandt 1992, Giusti 1980, 40–65; Jesseph 1989; Malet 1997a; Mancosu 1996, chap. 2; and Wallner 1903 all discuss various reactions to Cavalieri's method and the various changes in the fundamental concepts. For our purposes, the most significant feature of Wallis's reaction is his use of infinite sums of infinitely small parallelograms where Cavalieri had relied upon finite ratios of "all the lines" of one figure compared with another.

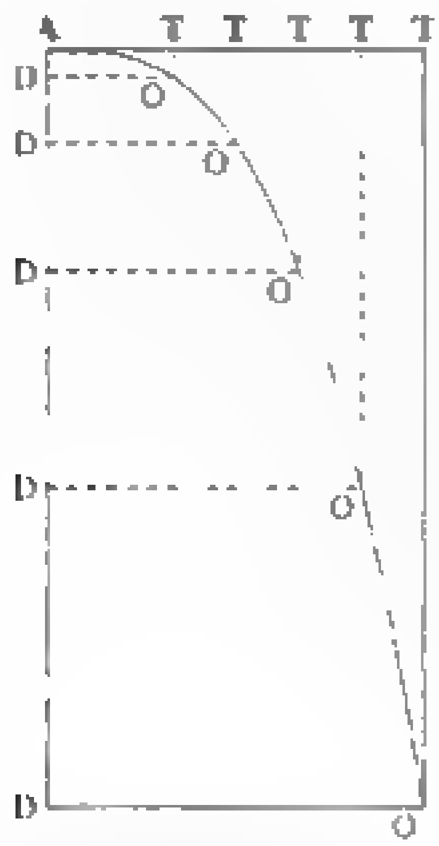

Figure 1.9

> If an infinite series is taken of quantities in triplicate ratio to a continually increasing arithmetical progression, beginning with 0 (or, equivalently, if a series of cube numbers is taken) this will be to the series of numbers equal to the greatest and equal in number as one to four. (*AI* 41; *OM* 1:382–83)

Given this result, Wallis turns to the quadrature of the cubic parabola in proposition 42, treating it as an infinite sum of lines forming a series of cubic quantities as in figure 1.9:

> And indeed let *AOT* (with diameter *AT*, and corresponding ordinates *TO*, *TO*, etc.) be the complement of the cubic semiparabola *AOD* (with diameter *AD* and corresponding ordinates *DO*, *DO*, etc.). Therefore (by proposition 45 of the *Treatise of Conic Sections*), the right lines *DO*, *DO*, etc. or their equals *AT*, *AT*, etc. are in subtriplicate ratio of the right lines *AD*, *AD*, etc. or their equals *TO*, *TO*, etc. And conversely these *TO*, *TO*, etc. are in triplicate ratio of the right lines *AT*, *AT*, etc. Therefore the whole figure *AOT* (consisting of the infinity of right lines *TO*, *TO*, etc. in triplicate ratio of the arithmetically proportional right lines *AT*, *AT*, etc.) will be to the parallelogram *TD* (consisting of just as many lines all equal to *TO*) as one to four. Which was to be shown. And consequently the semiparabola *AOD* (the residuum of the parallelogram) is to the parallelogram itself as one to four. (*AI* 42; *OM* 1:383)

Here, Wallis takes the figure as literally composed of indivisibles and unhesitatingly applies arithmetical principles to the solution of geometric problems. The "induction" that leads to his main arithmetical result is clearly not consonant with the classical standard of rigor, nor is his procedure of taking a ratio between two infinite series. But note also that his use of the method of indivisibles departs from the classical approach to geometry because it fails to observe the distinction between discrete and continuous magnitudes. In treating a continuous geometric figure as composed of sums of discrete points or lines, the method of indivisibles simply ignores the classical distinction. Wallis held that the method of indivisibles was essentially equivalent to the classical technique of exhaustion, although he admitted that it must be "applied with due caution" in order to avoid paradox.[59]

Where Wallis thought that the rigor of the method of indivisibles could be guaranteed by using it cautiously, others were convinced that the method was essentially ungeometrical and subjected it to searching criticisms. In particular, Paul Guldin attacked the method as ill-founded and unreliable, and his *Centrobaryca* (Guldin 1635–41) contained a long polemic against Cavalieri in which he argued that the method of indivisibles offends against the principle that there can be no ratio between infinities. As he observes, the attempt to find a ratio between "all the lines" of two figures can be understood only as an attempt to compare one infinite totality with another—but this is explicitly barred by the most basic canons of mathematical intelligibility.[60]

Guldin was not alone in his reservations about the method: others, including Galileo and the Jesuit mathematician André Tacquet, found fault with it and argued that it could lead to false results.[61] It is worth

59. In his *Mechanica*, Wallis declares "this doctrine of indivisibles (now everywhere accepted, and after Cavalieri, approved by the most celebrated mathematicians) replaces the continued adscription of figures of the ancients; for it is shorter, nor is it less demonstrative, if it is applied with due caution" (*Mechanica* 4; *OM* 1:646).

60. Guldin objects, "All the lines and all the planes of one and another figure are infinite and infinite; but there is no proportion or ratio of an infinite to an infinite. Therefore, etc. Both the major and minor premises are clear to all geometers, and so do not need many words" (1635–41, 4:341). Mancosu 1996, 50–64, is a detailed and enlightening study of Guldin's objections, which connects them to his broadly philosophical differences with Cavalieri over the epistemology of mathematics. Other studies of Guldin's attack on Cavalieri are Andersen 1985, sec. 10, and Giusti 1980, sec. 3.

61. Galileo's objections appear largely in letters to Cavalieri, which are studied in Andersen 1985, Giusti 1980, and De Gandt 1991. Tacquet's objection is contained in his 1651 work *Cylindrica et Annularia* (Tacquet 1669, 3:38–39). Mancosu 1996, 119–29, places these and similar objections in their mathematical and philosophical context.

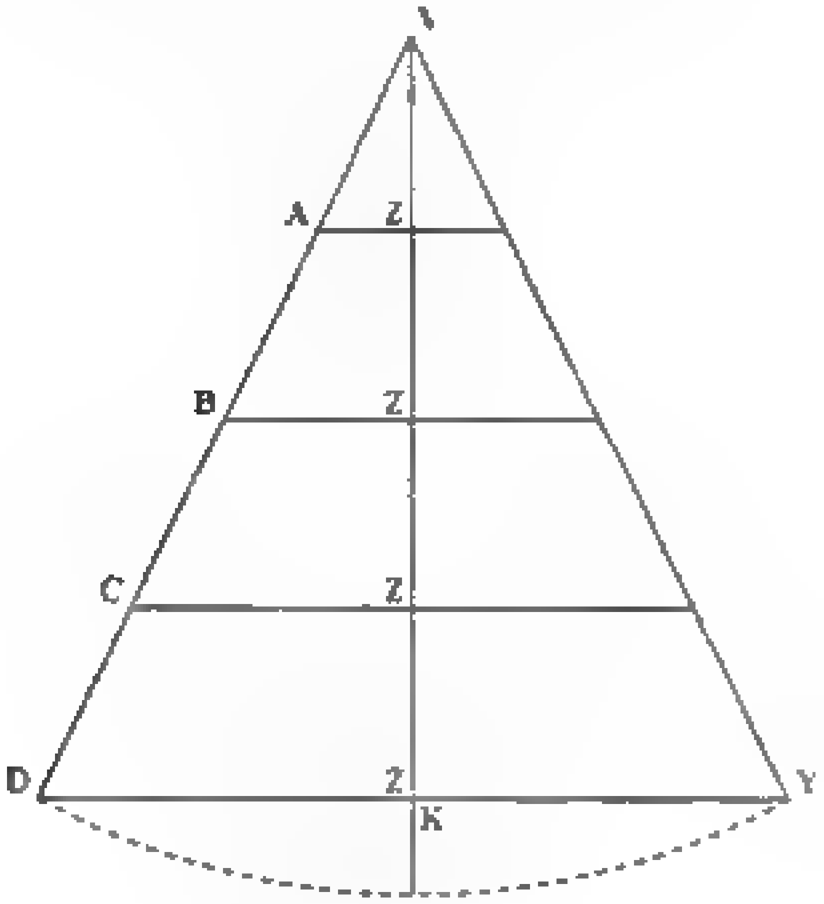

Figure 1.10

considering Tacquet's objection briefly, as it (and attempts to overcome it) brings out some of the significant tensions in the theory of indivisibles. Tacquet argues that, according to the method of indivisibles, the cone *DVY* (in figure 1.10) will have a surface area equal to that of a circle with radius *KD*, since the area can be thought of as consisting of all the peripheries of the circles with radii *ZA, ZB, ZD*, etc. This yields πKD^2 for the surface area, contrary to the established result of $\pi KD \cdot \sqrt{KD^2+KV^2}$. Barrow took up this objection in the second of his *Geometrical Lectures*, and his reaction to it is intriguing:

> I respond that this is the wrong way to undertake the calculation, for in computing the peripheries of which the [curved] surface consists we must use a different way of reasoning from that by which the lines making a plane surface are computed, or the planes from which a body is formed. That is, the number of peripheries constituting the curved surface produced by the revolution of the line *VD* must be judged from the multitude of points which are in this generative line itself. This is because only a single periphery passes through each point on the line, nor could any more pass through it, whatever the axis may be, or whether the line is further from the axis or nearer to it, for it is the axis alone, according to its various remoteness or nearness and vary-

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My treatment of these sorts of difficulties with the theory of indivisibles is correspondingly brief.

> ing position which determines the magnitude of the said peripheries. But the multitude of lines of which the plane *DVK* is supposed to consist, and the multitude of planes of which the solid *DVY* consists, is to be judged by the number of points in the axis *VK*; nor indeed can there be more parallel right lines perpendicular to *VK* and contained within the limits *VK*, or can more such parallel planes be drawn, than are equal in number to the multitude of those points [in the axis *VK*]. In observing this difference (which is to be carefully minded) we shall avoid all error *and I judge we shall search out the surfaces generated by the rotation of such curves by the simplest method that the nature of these things admits.* (Barrow 1860, 2:183–84)

The oddity of this response is evident, since it commits Barrow to the absurd thesis that the line *VK* must contain fewer points than the line *VD* because *VK* is shorter than *VD*, i.e., he contends that the number of points on a line is a function of its length.

Concern about the foundations of the method led Barrow to some further torturous reasoning intended to show that although "there is no greater fault in geometry than to seek or assert the ratio of heterogeneous quantities to one another" (*LM* 16, 260) the method of indivisibles does not actually offend against this principle. He explains

> It is true that among those who in the solutions of problems or demonstrations of theorems apply the method of indivisibles, such expressions often occur. All of these parallel lines are equal to such a plane, the sum of these parallel planes constitutes a solid; but they explain their meaning and say that by lines the understand nothing other than certain parallelograms with a very small and (pardon the expression) inconsiderable altitude; by planes they understand likewise nothing but prisms or cylinders of an altitude not to be computed. Or at least by the sum of lines and planes they do not indicate a certain finite and determinate sum, but an infinite or indefinite sum equal to the number of points on some right line. (*LM* 16, 260–61)

Barrow then asserts that anyone who actually claimed that such a sum of "inferior homogeneals" could compose an object of heterogeneous nature "could easily be convinced of his error." Notwithstanding Barrow's assertions of the fidelity of the method to the classical criteria of rigor, one of the key projects in seventeenth-century mathematics was

to sort out the conditions under which the method of indivisibles could be used, and ultimately to render it as secure as classical methods.

Hobbes was familiar both with the method and the controversies surrounding it, and as we will see in chapter 3, his own approach to the problem of quadrature is strongly influenced by the method of indivisibles. Hobbes also criticized Wallis's presentation of the method of indivisibles as methodologically suspect, while arguing that only his own metaphysics offered the hope of securely grounding the method and of introducing powerful new mathematical techniques that would enable the solution of any mathematical problem. Before turning to an exposition of Hobbes's philosophy of mathematics, we must first examine the broader social and philosophical background to the dispute, which is the subject of the next chapter.

CHAPTER TWO

The Reform of Mathematics and of the Universities

Ideological Origins of the Dispute

> For seeing the Universities are the Fountains of Civill, and Moral Doctrine, from which the Preachers, and the Gentry, drawing such water as they find, use to sprinkle the same (both from the Pulpit, and in their Conversation) upon the People, there ought certainly to be great care taken, to have it pure, both from the Venime of Heathen Politicians, and from the Incantation of Deceiving Spirits.
>
> —Hobbes, *Leviathan*

As we have seen, Hobbes prided himself upon having reformed some of the principles of mathematics. But his reforming spirit was hardly confined to this subject, for his philosophy was intended to transform social institutions as well. Indeed, in an age when the rhetoric of reformation appears throughout scientific, theological, and philosophical literature, Hobbes stands out as one of the most committed reformers. The aim of the present chapter is to explain how Hobbes's account of the role of the university in a well-ordered commonwealth brought him into conflict with such notable university men as Ward and Wallis. This undertaking will help to uncover the origins of the dispute between Hobbes and Wallis, and it can also help explain at least some of the vehemence and doggedness with which the dispute was conducted. I begin with an overview of anti-Hobbes opinion provoked by the publication of *Leviathan* and then proceed to a brief summary of Hobbes's doctrines as they relate to the role of the universities. In section three I explore how Hobbes's views on this topic are related to other seventeenth-century critiques of the university system. With this background in hand, I then argue that the original impetus to the quarrel, and much of the fervor with which it was conducted, can be accounted for by seeing Wallis as a defender of the university in a contemporary debate over the status of universities in the commonwealth.

2.1 HOBBES AND HIS ENEMIES

The publication of *Leviathan* in 1651 made Hobbes's name anathema to much of the learned public in England. Where he had previously enjoyed a considerable reputation as a man of letters, the doctrines set forth in his masterwork aroused a storm of opposition, and critiques of Hobbes and his principles remained a fixture in English letters until well after his death. Even some who had previously held Hobbes in high regard were so repulsed by what they found in *Leviathan* that they took pains to dissociate themselves from the name of the notorious Monster of Malmesbury. Seth Ward is a notable case in point.

Ward was an important figure in mid-seventeenth-century British science, having been appointed to Oxford's Savilian Chair of Astronomy in 1649 at the age of thirty-two.[1] Indeed, the same Parliamentary board of visitors that intruded Wallis into the Savilian Professorship of Geometry was prepared to place Ward into the other Savilian chair, notwithstanding his known Royalist and Anglican sympathies.[2] The appointment naturally required Ward to profess allegiance to the new Commonwealth and abjure his allegiance to the Church of England—the same church in which he would acquire the rank of bishop after the restoration of Charles II. There was nothing unique in Ward's manifestly cynical attitude, and few seem to have held it against him. He was prominent in the intellectual life of Oxford in the 1650s, particularly in his promotion of scientific and mathematical learning. Together with Wallis and John Wilkins (who was named warden of Wadham College in 1649), Ward was instrumental in bringing the new "mechanical philosophy" to Oxford, and it is largely through the efforts of their circle that Oxford could claim a place of importance in the scientific world of the day (Shapiro 1969). Sir Kenelm Digby could refer without irony to Ward, Wallis, and Wilkins as exercising a "worthy Triumvirate . . . in literature and all that is worthy,"[3] and these three were without question among the most important scientific figures at Oxford in the 1650s.

1. For biographical details on Ward, see Pope 1697 and Fletcher 1940; the entries in the *Dictionary of Scientific Biography* and *Dictionary of National Biography* are also useful.

2. In fact, Ward had been deprived of his fellowship at Cambridge in 1644 for collaborating in the publication of an anti-Puritan tract, a project in which he assisted Isaac Barrow.

3. The reference is in a letter from Digby to Wallis, dated 1/11 August 1657 and published as part of Wallis's *Commercium Epistolicum de Quaestionibus quibusdam Mathematicis nuper habitum* (Wallis 1658, 10–12).

It is in his capacity as a promoter of scientific learning that Ward first came into contact with Hobbes, and it seems that he actively supported the publication of Hobbes's treatise *Humane Nature* in 1650. This work contains the first thirteen chapters of the treatise later published as *The Elements of Law* and deals with the general principles of scientific investigation, particularly as applied to human psychology.[4] Ward apparently contributed the "publisher's preface" to the book, in which he commends Hobbes for having "written a body of philosophy, upon such principles and in such order as are used by men conversant in demonstration" (*EL* preface; *EW* 4:xi).[5] In 1651, upon Hobbes's return to England, Ward also made a special trip to London to make his acquaintance, and there is no doubt that he regarded him as an important contributor to the advancement of science and philosophy.

Ward was concerned with more than the advancement of science during his years at Oxford, however. He pursued his considerable theological interests as well, taking the degree of doctor of divinity from Oxford in 1654. It is on theological issues that he initially broke with Hobbes, as is evident from the preface to his 1652 *Philosophicall Essay Towards an Eviction of the Being and Attributes of God, the Immortality of the Souls of Men, The Truth and Authority of Scripture.*[6] Speaking of himself in the third person, Ward declares

> He must needs acknowledge, that before the edition of this he hath seen M. Hobs his Leviathan, and other Bookes of his, wherein that which is in this Treatise intended as the main Foundation whereon the second Discourse (Of the Souls Immortality) insists, is said to imply a contradiction, viz. That there are any

4. The full title, as published in 1650 reads *Humane Nature, Or: The fundamental Elements of Policie. Being a Discoverie of the Faculties, Acts, and Passions of the Soul of Man, from their original causes; According to such Philosophical Principles as are not commonly known or asserted.*

5. The grounds for attributing the preface to Ward are strong, but not completely certain. Anthony à Wood reports that "Seth Ward writ the epistle to the reader in the name of Francis Bowman bookseller, before this book" (Wood [1813] 1967, 3:column 1209). Hobbes himself shared this opinion, and in addressing Ward in the *Six Lessons* remarked that "Which [part of my book that concerneth policy merely civil], if you, the Astronomer, that now think the Doctrine unworthy to be taught, were pleased once to honour with praises printed before it, you are not very constant nor ingenuous. But whether you did so or not, I am not certain, though it was told me for certain" (*SL* 6; *EW* 7:336).

6. The role of Ward's *Philosophicall Essay* in Hobbes's controversy with Ward and Wallis has been amply investigated by Probst (1993); my account of the matter differs minimally from his.

> such things as Immateriall or Incorporeall substances. Upon which occasion he thought good onely to say, That he hath a very great respect and a very high esteem for that worthy Gentleman, but he must ingenuously acknowledge that a great proportion of it is founded upon a belief & expectation concerning him, a belief of much knowledge in him, and an expectation of those Philosophicall and Mathematicall works, which he hath undertaken; and not so much upon what he hath yet published to the world, and that he doth not see reason from thence to recede from any thing upon his Authority, although he shall avouch his discourse to proceed Mathematically. That he is sure he hath much injured the Mathematicks, and the very name of Demonstration, by bestowing upon it some of his discourses, which are exceedingly short of that evidence and truth which is required to make a discourse able to bear that reputation. (Ward 1652, preface, A3)

It is unclear which "other Bookes" besides *Leviathan* Ward may have had in mind in writing this passage, but the general message is clear enough: the radical materialism of *Leviathan* is anathema to him, and this principally because it does not allow for the immortality of the soul.[7]

It is worth pointing out that *Humane Nature* (which Ward had seen and praised) differs from *Leviathan* in not explicitly denying the existence of immaterial substances. Hobbes's earlier work gives a mechanistic account of human sensation and argues for the subjectivity of all sensory qualities, concluding that "*whatsoever accidents* or qualities our senses make us think there be in the *world*, they be *not* there, but are *seemings* and *apparitions* only: the things that really *are* in the world without us, are those *motions* by which these seemings are caused" (*EL* 1.2.10; *EW* 4:8). But *Humane Nature* does not assert that the human mind is itself an assemblage of bodies in motion, and Hobbes even seems to leave room for an immaterialistic conception of the mind in the first chapter when he distinguishes faculties of the body from faculties of the mind. As a matter of fact, other authors of the period (such as Descartes) could combine a mechanistic treatment of sensation with a thoroughly immaterialistic understanding of the hu-

7. Ward summarizes his principal argument for immortality as follows: "Now the substance of all that I shall speak towards the demonstration of the souls Immortality shall be summarily comprised in this one syllogism. Whatsoever is incorporeall is immortal. But the souls of men are incorporeall substances, *Ergò*" (Ward 1652, 35).

man soul or intellect, and before the publication of *Leviathan* Ward was presumably unaware of the extent of Hobbes's commitment to materialism.[8]

The materialism of *Leviathan* was attacked by others besides Ward, and the polemics against Hobbesian materialism were mounted principally for theological reasons.[9] By denying the coherence of the concept of an immaterial substance, Hobbes challenged the very foundation of traditional theology in which souls, angels, and God himself were conceived of as essentially immaterial beings. Aubrey put the matter succinctly with the report that "the divines say, 'Deny spirits, and you are an atheist'" (Aubrey 1898, 2:318). In rejecting the standard opposition between the material and the spiritual, Hobbes therefore set himself the task of providing an alternative reading of scriptural passages dealing with such (allegedly immaterial) beings as spirits or angels. This led to one of the most controversial parts of *Leviathan*—chapter 34, which treats "*Of the Signification of* SPIRIT, ANGEL, *and* INSPIRATION *in the Books of Holy Scripture*." The chapter attempts to show that all biblical references to spirits or angels can be understood materialistically, and this is one of the parts of *Leviathan* that Hobbes admitted "may most offend" because it contains "certain Texts of Holy Scripture, alleged by me to other purpose than ordinarily they use to be by others" (*L* epistle; *EW* 3:ii).

I will investigate Hobbes's theological views, and particularly his

8. This suspicion is strengthened by Ward's remark in the preface to the *Philosophicall Essay* that Hobbes's denial of immaterial spirits

> *can rationally import no more but this, That he himself hath not an apprehension of any such beings, and that his cogitation (as to the simple object of it) hath never risen beyond imagination, or the first apprehensions of bodies performed in the brain; but to imagine that no man hath an apprehension of the God-head, because he may not perhaps think of him so much as to strip off the corporeal circumstances wherewith he doth use to fancy him; or to conclude every man under the sentence of being non-sensicall, whosoever have spoken or written of Incorporeal substances, he doth conceive to be things not to be made good by the Authority of* M. Hobbs. (Ward 1652, preface, sig. A4)

The opposition here between imagination on the one hand (which depends upon sensation, which itself is explained materialistically in terms of motions in the brain) and intellectual apprehension on the other (which "strips off" corporeal circumstances and enables the conception of immaterial beings) suggests that Ward accepted a broadly Cartesian picture of the mind, where the "lower" faculties depend upon the body, while the "higher" intellectual faculties are immaterial. I will have more to say about these matters as they relate to Hobbes's account of reasoning in chapter 5.

9. See Mintz 1962, chaps. 4–5, for a summary of the antimaterialist and theological attacks on Hobbes. I will return to this issue in chapter 7.

alleged atheism, in greater detail in chapter 7, but it is worthwhile stressing the point that his contemporaries saw a seamless connection between materialistic metaphysics, the denial of an immortal soul, a moral philosophy of pure egoism, and atheism. Hobbesian philosophy was therefore easily caricatured as the creed of the godless libertine who denies the possibility of an afterlife and takes the satisfaction of selfish desires as the summum bonum. Moreover, Hobbes's frequent claims to have grounded his philosophy in demonstration and his characterization of alternative systems as nonsensical Scholastic jargon gave him a reputation for dogmatism.

This picture of Hobbes as a dogmatic atheist and libertine emerges quite clearly in a letter from Wallis to Thomas Tenison (later archbishop of Canterbury) written approximately a year after Hobbes's death (the letter is dated 30 November/10 December 1680). Tenison had attacked Hobbes's philosophy with his 1670 book *The Creed of Mr. Hobbes Examined*, and he was evidently planning to write a life of Hobbes that would show the erroneous nature and harmful consequences of Hobbes's philosophy and thus serve as an admonition to all who might be tempted to embrace Hobbesian principles and thereby fall into impiety. In the letter, Wallis approves Tenison's project and offers his own opinion of Hobbes and his philosophy. He claims to have had no personal acquaintance with Hobbes, but that as a result of their controversy Wallis formed the firm opinion that the Monster of Malmesbury "was not a man of strong Reason; but only of a bold daring phansy, which, with his magisteriall way of speaking, did (not convince, but) sway those that loved to be Atheists, and were glad to have any body dare boldly to say, what they wish'd to be true" (Bodleian Library MS. Add. D.105, f. 70^r). Wallis concludes that Hobbes's principles must be those of a man who secretly dreads the punishments of the afterlife even as he denies the very existence of an afterlife:

> In summe; I can hardly believe y^t Mr Hobbes himself (nor perhaps any other pretenders to it) was so much an Atheist, as he would fain have been: but did really dread a future state: otherwise, he would not have been so dreadfully afraid of death, as y^e concurrent testimony of those who knew him do represent him. (Bodleian Library MS. Add. D.105, f. 70^v)

This morbid fear of death was, according to Wallis, confirmed by an incident related to him "divers years agoe," in which "a great Lady" reported that Hobbes had declared "if he were master of all y^t was to dispose of, he would give it to live one day." To the reproach that "a

person of his knowledge, & who had so many friends to oblige or gratify would not deny himself one days content of things if thereby to be able to gratify them with all y^e World," Hobbes is supposed to have evinced both his utter selfishness and his terror of death with the reply "Madame, what should I be the better for that, when I am dead? I say again If I had all y^e World to dispose of, I would give it to have one day" (Bodleian Library, MS. Add. D.105, 70^v).

Wallis found further evidence for this view of Hobbes and his philosophy "in y^t sermon at ye Funeral of ye late Earl of Rochester: Who could talk Atheisticall things with as much briskness & as much wit as Mr Hobbes, and trip most at sense & reason: yet could not thoroughly beleeve it; but was galled (we are to understand) with a recoiling conscience" (Bodleian Library, MS. Add. D.105, 70^v). In addition to his acknowledged literary gifts Rochester was one of the most notorious libertines associated with the court of Charles II (a venue not known for its modest living). However, shortly before his death he embraced Christianity and repented his former life of dissolution.[10] The sermon preached by Robert Parsons at the funeral of Rochester singled out Hobbes as the man whose principles had done the most to encourage the life of atheistic debauchery. Speaking of the departed Rochester, Parsons declared:

> How remarkable was his Faith, in a hearty embracing and devout confession of all the Articles of our Christian Religion, and all the Divine mysteries of the Gospel? Saying, that *that absurd and foolish Philosophy, which the world so much admired, propagated by the late Mr.* Hobbs, *and others, had undone him, and many more of the best parts in the Nation;* who, without Gods great mercy to them, may never, I believe, attain such a Repentance. (Parsons 1680, 26)

We need not presume that Hobbes's enemies had the correct interpretation of his philosophy, its consequences, or its author's opinions on matters of morals or the afterlife. Nevertheless, it is clear that he had acquired the reputation as an atheist, a libertine, and the proponent of views that were inimical to religion and morals.

Quite aside from the thoroughgoing materialism that led many to regard it with horror, *Leviathan* was also controversial for its political

10. On Rochester, his conversion, and his relationship to Hobbes, see Mintz 1962, 140–42.

doctrines. The central theme in Hobbes's political theory is the necessity of absolute, undivided sovereignty. Sovereign power is not to be shared among competing persons or institutions, for (on Hobbes's analysis) such divided power must inevitably produce conflicting claims upon the loyalties of subjects, and these conflicts must eventually lead to civil war. Similarly, sovereignty cannot be limited to a particular area of competence, because the imposition of limits on a sovereign's power supposes the existence of a supreme authority over and above that of the sovereign. One important consequence of this theory is that the civil and ecclesiastical sovereign are "consolidated" in one supreme authority, and such sovereigns "have all manner of Power over their subjects, that can be given to a man, for the government of mens externall actions, both in Policy and Religion; and may make such Laws, as themselves shall judge fittest, for the government of their own Subjects, both as they are the Common-wealth, and as they are the Church: for both State, and Church are the same men" (*L* 3.42, 299; *EW* 3:546) It is the sovereign, and only the sovereign, who may lawfully resolve questions of church governance, scriptural interpretation, liturgy, and other ecclesiastical matters.

Although such a doctrine could find favor among committed Royalists who desired to see the crown's power absolute and unchallenged, it was bitterly opposed by those who insisted that spiritual matters must be entrusted to ecclesiastical authorities distinct from the civil sovereign. These included Anglicans who favored an episcopal system granting broad powers to bishops, Presbyterians who would abolish episcopacy and invest ecclesiastical authority in a body of clergymen and lay elders, and other varieties of Protestant thought that denied the authority of the civil sovereign in religious matters. Hobbes provoked the wrath of these churchmen with his version of ecclesiastical history in part 4 of *Leviathan*. He there describes the clergy as part of the "Kingdome of Darknesse" that through various deceitful practices has conspired to usurp the civil power. The Catholic Church is the most notable agent of darkness in this Hobbesian demonology, but Anglican bishops and Presbyterian doctors of divinity are also condemned for their attempt to establish a power contrary to that of the rightful sovereign. In a memorable passage in chapter 47 of *Leviathan* Hobbes describes the sovereign as having been bound by knots tied by ambitious clergymen, which (at least in England) have finally been untied by a continuing process of reformation that leads to "the dissolution of the praeterpoliticall Church Government in England":

> First, the Power of the Popes was dissolved totally by Queen Elizabeth; and the Bishops, who before exercised their Functions in Right of the Pope, did afterwards exercise the same in Right of the Queen and her Successours; though by retaining the phrase of *Jure Divino*, they were thought to demand it by immediate Right from God: And so was untyed the first knot. After this, the Presbyterians lately in England obtained the putting down of Episcopacy: and so was the second knot dissolved: And almost at the same time, the Power was taken also from the Presbyterians: And so we are reduced to the Independency of the Primitive Christians to follow Paul, or Cephas, or Apollos, every man as he liketh best: Which, if it be without contention, and without measuring the Doctrine of Christ by our affection to the Person of his Minister . . . is perhaps the best. (*L* 4.47, 385; *EW* 3:696)

This endorsement of "Independency" may seem surprising coming from a man who holds that the civil sovereign has the absolute power to compel conformity in all matters of worship. The Independents and the Presbyterians were the two principal factions among supporters of the Parliamentary cause in the English Civil War. They were united in opposition to the Anglican system of episcopacy, but disagreed about what system should replace it. As the name suggests, the Independents favored a relatively loose form of church governance in which local congregations could resolve liturgical or doctrinal issues independently, while the Presbyterians insisted upon the imposition of a strict system of church organization that would enforce conformity throughout the nation. Hobbes's suggestion that Independency is "perhaps the best" reflects, in part, his concern that any centralized system of church government could pose a threat to the sovereign's power. Moreover, the events of the Civil War clearly suggested that, although the sovereign has the right to impose conformity in matters of worship, it is not always in his interest to enforce a policy of strict conformity. A wise sovereign could best avoid potential challenges to his rule by permitting a fairly wide latitude in belief and worship, reserving his intervention in religious affairs to discipline those who claim a power in opposition to his own.[11]

11. Richard Tuck, in commenting upon this aspect of Hobbes's theory notes that "in Hobbes's eyes, the sovereign would not have the same kind of reasons for enforcing particular dogmas upon his citizens as churches historically had acted on in controlling their members. His right to enforce doctrine was essentially negative, and intended above all to stop *non-sovereigns* from claiming such a right" (Tuck 1989, 88). For more on

It should by now be evident that the publication of *Leviathan* had the remarkable effect of offending nearly every side in the struggles taking place in England in 1651. This is not to say that Hobbes was entirely without a readership or admirers, but he clearly had done enough to guarantee that he would have a plentiful supply of enemies. One of the most important sources for opposition to Hobbes was the universities, and particularly the University of Oxford. It is therefore advisable that we turn our attention more closely to Hobbes's doctrines as they relate to the universities in order to understand the background to his war with Wallis.

2.2 HOBBES AND THE UNIVERSITIES

In holding that all institutions in a commonwealth must be under the absolute control of the sovereign, Hobbes thereby requires that the university be subordinated to the state. No doctrines may be legitimately taught, except those explicitly permitted by the authority of the sovereign, and the sovereign is the sole judge of which teachings are permissible (*L* 2.18, 91; *EW* 3:164–65). This principle extends, ironically enough, to the teaching of geometry, for Hobbes holds that the sovereign may ban the teaching of geometry and order "the burning of all books of Geometry" if he should judge any geometric result contrary to his right of dominion.[12]

Hobbes directed some of his most vitriolic polemics at the universities, largely because he saw them as endangering civil peace by challenging the authority of the sovereign. Although some parts of his critique of the universities were intended to apply to Continental as well as English universities, it is clear that Oxford and Cambridge were his principal targets. He condemns the universities as agents of the "Kingdome of Darknesse" and traces their history from the schools of the ancient Jews and Greeks. In his portrayal, the "unprofitable" schools of ancient times have been succeeded by equally useless and benighted universities, which are so far from being centers of learning that they

Hobbes's account of church, state, and religious order see Sommerville 1992, chaps. 5 and 6.

12. Hobbes declares, "For I doubt not, but that if it had been a thing contrary to any mans right of dominion, or to the interest of men that have dominion, *That the three Angles of a Triangle should be equall to two Angles of a Square;* that doctrine should have been, if not disputed, yet by the burning of all books of Geometry, suppressed, as farre as he whom it concerned was able" (*L* 1.11, 50; *EW* 3:91).

teach neither genuine philosophy nor (until very lately) even the rudiments of geometry. A university, Hobbes declares, is

> a Joyning together, and an Incorporation under one Government of many Publique Schools, in one and the same Town or City. In which, the principall Schools were ordained for the three Professions, that is to say, of the Romane Religion, of the Romane Law, and the Art of Medicine. And for the study of Philosophy it hath no otherwise place, then as a handmaid to the Romane Religion: And since the Authority of Aristotle is onely current there, that study is not properly Philosophy, (the nature whereof dependeth not on Authors,) but Aristotelity. And for Geometry, till of very late times it had no place at all; as being subservient to nothing but rigide Truth. And if any man by the ingenuity of his owne nature, had attained to any degree of perfection therein, hee was commonly thought a Magician, and his Art Diabolicall. (*L* 4.46, 370; *EW* 3:670)

The charges Hobbes brings against the universities are grave indeed, but three are of particular concern in this context. These include: obscurantism in teaching nonsensical scholastic metaphysics, the fomenting of civil war by training preachers in theological doctrines that undermine the sovereign power, and the promotion of civil discord by teaching men that the lawful government is an illegal tyranny.

To understand these charges properly we must first observe that Hobbes regarded the university as an institution of central importance in maintaining civil order. He was acutely aware that a subject's loyalty to his sovereign is not an inborn trait, but rather something that must be carefully inculcated by an aggressive system of education. By nature people desire power for themselves, and their pride or self-conceit leads them to resent any authority placed over them. For any system of social organization to work, people must therefore be brought to see that their interests lie in submission to an all-powerful central authority, and their natural rebelliousness must be curbed by the threat of sanction from the sovereign. It is therefore only through the combination of education and discipline that the fundamental problem of the state of nature can be solved on a continuing basis.

The alternative to civil peace is civil war, and Hobbes traces the root of civil war to the prevalence of private judgment, and especially private judgment about matters of political or religious obligation. In the state of nature, conflicting private judgments abound and are the source of the war of all against all. Hobbes holds that "so long as every

man holdeth this Right of doing any thing he liketh; so long are all men in the condition of Warre" (*L* 1.14, 65; *EW* 3:118) and "if their actions be directed according to their particular judgements, and particular appetites, [men] can expect thereby no defence, nor protection, neither against a Common enemy, nor against the injuries of one another" (*L* 2.17, 86; *EW* 3:155). But even in the state of society there is a danger of lapsing back into chaos if citizens are allowed to determine for themselves what is to count as lawful, or which obligations they have. Hobbes lists several "Opinions, contrary to the peace of Man-kind," that tend toward civil war, including:

> That men shall Judge of what is lawfull and unlawfull, not by the Law it selfe, but by their own Consciences; that is to say, by their own private Judgements: That Subjects sinne in obeying the Commands of the Common-wealth, unlesse they themselves have first judged them to be lawful: That their Propriety in their riches is such, as to exclude the Dominion, which the Common-wealth hath over the same; That it is lawfull for Subjects to kill such, as they Call Tyrants; That the Soveraign Power may be divided, and the like. (*L* 2.30, 179; *EW* 3:330–31)

The task of the sovereign therefore includes that of keeping private judgment in check. But this can be accomplished only by getting citizens to relinquish their dangerous opinions and to accept the authority of the sovereign, and Hobbes sees the universities as the principal tool by which this is to be effected.

As Lloyd has noted, Hobbes "holds a decidedly trickle-down theory of education and influence," in which principles taught at the universities will be accepted by clergymen trained there and thus disseminated throughout the society (Lloyd 1992, 193–94). Hobbes himself declares it "manifest, that the Instruction of the people, dependeth wholly, upon the right teaching of Youth in the Universities" (*L* 2.30, 180; *EW* 3:331). His approach to political theory consequently requires that one primary use of the sovereign's power is to coerce the universities into reforming their principles so that the doctrines taught there are conducive to social stability. Not surprisingly, Hobbes held out the hope that "at one time or other, this writing of mine, may fall into the hands of a Soveraign, who will consider it himselfe . . . without the help of any interested or envious Interpreter; and by the exercise of entire Soveraignty, in protecting the Publique teaching of it, convert this Truth of Speculation, into the Utility of Practise" (*L* 2.31, 194; *EW* 3:358). In particular, Hobbes hoped that such an enlightened sovereign might see

in *Leviathan* the ideal text to be assigned to students in the universities. Thus, in the "Review and Conclusion" to *Leviathan* he observed, "I think [this book] may be profitably printed, and more profitably taught in the Universities, in case they also think so, to whom the judgement of the same belongeth. For seeing the Universities are the Fountains of Civill, and Morall Doctrine, . . . there ought certainly to be great care taken, to have [these fountains] pure, both from the Venime of Heathen Politicians, and from the Incantation of Deceiving Spirits" (*L* review and conclusion, 395; *EW* 3:713). This declaration became the basis for Wallis's repeated accusations that, as an enemy of the universities and a vain dogmatist in love with his own theories, Hobbes intended to have the doctrines of *Leviathan* set up as official teaching at Oxford and Cambridge.

The specific errors with which Hobbes charges the universities are closely related to his conception of the university as an institution crucial for the maintenance of civil peace. Subjects can be misled by prideful and ambitious teachers into accepting doctrines that work against social order, even if the connection between the false doctrine and the resulting social instability is not immediately apparent. Aristotelian metaphysics is a case in point. According to Hobbes, the Scholastic presentation of Aristotelian hylomorphism and the doctrine of separated essences are mistaken doctrines with politically dangerous consequences. Those who are taught these doctrines are led to think that a person's soul can exist apart from the body, that bread and wine can be transubstantiated into flesh and blood, and that virtues can exist separated or apart from humans and be deposited or "blown in" to the soul. These and other doctrines derived from Aristotelian teachings,

> serve to lessen the dependance of subjects on the soveraign power of their countrey. For who will endeavour to obey the laws if he expect obedience to be powered or blown into him? Or who will not obey a priest, that can make God, rather than his soveraign; nay than God himselfe? Or who, that is in fear of ghosts, will not bear great respect to those that can make the holy water that drives them from him? (*L* 4.46, 373; *EW* 3:675)

There is nothing novel in Hobbes's claim that the universities are intellectual backwaters corrupted by the undue authority of Aristotle. Condemnations of Aristotle and the alleged prevalence of Aristotelianism at the universities are a commonplace among seventeenth-century "moderns" such as Descartes, Bacon, and Galileo. But Hobbes is cer-

tainly unique in seeing Scholastic obscurantism as productive of social disorder, rather than simply confusion, error, or bad philosophy.

Hobbes's attacks on the seditious doctrines promulgated by the clergy are related to this condemnation of Aristotelianism in the universities. The Catholic doctrine of the pope's supreme ecclesiastical power was naturally anathema to Hobbes, but he equally opposed the Presbyterians' insistence that temporal authority must be subservient to spiritual authority. According to Hobbes, Catholics and Presbyterians share a common error of believing that the kingdom of God is present in the form of the church, and consequently that the civil sovereign is subject to the dictates of the church.[13] As he puts it, "The greatest, and main abuse of Scripture, and to which almost all the rest are either consequent, or subservient, is the wresting of it, to prove that the Kingdome of God, mentioned so often in the Scripture, is the present Church" (*L* 4.44, 334; *EW* 3:605). The principal effect of this error is to undermine the sovereign's power by dividing it: if the kingdom of God is the present church, the temporal authority of the sovereign is placed in opposition to the spiritual authority of the church, and power rightfully belonging to the civil sovereign is usurped by divines. The universities, as centers of religious authority and teaching, stand naturally to benefit from this usurpation of civil power, and they endeavor to protect their position by employing "the frivolous Distinctions, barbarous Terms, and obscure Language of the Schoolmen," in order to "keep these Errors from being detected, and to make men mistake the *Ignis fatuus* of Vain Philosophy, for the Light of the Gospel" (*L* 4.47, 383; *EW* 3:693).

Hobbes's third principal indictment of the universities is that they have taught men "to call all manner of Common-wealths but the Popular . . . *Tyranny*" (*L* 4.46, 377; *EW* 3:682). Here again, he detects the corrupting influence of Aristotle, combined with other "heathen politicians" such as Cicero. The teaching of their false doctrines of right and obligation has resulted in a number of dangerous errors, not

13. Lloyd summarizes this aspect of Hobbes's thought nicely: "Indeed, for Hobbes, Presbyterians and papists are reverse sides of the same coin: Both contend that the kingdom of God presently exists; both counterpose temporal and spiritual government, asserting the latter to have primacy; both engage in what Hobbes sees as esoteric and dangerous doctrinal disputes; and each takes itself to be the only true church. In Hobbes's view, excepting their commendable attack on popish superstition, Presbyterians are just as bad as Catholics—and much more dangerous given English commitments, because much more influential" (Lloyd 1992, 196).

least of which is the opinion "that in a wel ordered Common-wealth, not Men should govern, but the Laws." From these false tenets, the ambitious doctors in the universities "induce men, as oft as they like not their Governours, to adhaere to those that call them Tyrants, and to think it lawfull to raise warre against them" (*L* 4.46, 377–78; *EW* 3:683–84). Hobbes thus takes the English Civil War to be a university-instigated conflict, and one principally caused by an ambitious Presbyterian clergy, whose pursuit of spiritual and temporal power led to disaster. But once the causes of the rebellion have been uncovered, civil peace can be ensured by reforming the universities and subordinating them to the power of the lawful sovereign.

This is all well and good, but one might ask what bearing it has on Wallis and the quadrature of the circle. For present purposes it is sufficient to note that, in addition to holding a position of considerable power within the University of Oxford, Wallis was a prominent Presbyterian who served as a secretary to the Westminster Assembly of Divines, and that his appointment as Savilian professor of geometry in 1649 was effected by the victorious Parliamentary forces who had recently executed Charles I. Indeed, he was exactly the kind of man Hobbes detested: an ambitious doctor of divinity with close ties to the Puritan establishment. In the *Six Lessons,* Hobbes could complain that his political doctrine "is generally received by all but those of the Clergy, who think their interest concerned in being made subordinate to the Civil Power; whose testimonies therefore are invalide," and insist that because "so much as could be contributed, to the Peace of our Country, and the settlement of Soveraign Power without any Army, must proceed from Teaching; I had reason to wish, that Civill Doctrine were truly taught in the Universities" (*SL* 6; *EW* 7:333, 335). There were few, if any, in England who had more reason to fear and detest Hobbes and his teachings, and Wallis was not a man to work quietly behind the scenes to defend himself and his position against a perceived threat.

2.3 THE UNIVERSITY REFORM DEBATE

In calling for a fundamental reform of the universities, Hobbes took part in an important seventeenth-century debate over the status of universities in England. Many Puritan critics of Oxford and Cambridge had also campaigned for an overhaul of the universities, with the intent of making their practices consonant with the principles of true government and religion. It was widely agreed both by the critics and defend-

ers of the universities that their central function was to provide ministers for the state church, and Puritan calls for reform were generally intended to ensure that these parsons would be staunch defenders of the newly reformed state church. With the Parliamentary party's victory in the Civil War and the abolition of monarchy, the universities were purged of the most committed Anglicans and Royalists and left largely in the hands of the two main Puritan groups: the Presbyterians and the Independents. In this process, Wallis was appointed Savilian professor of geometry at Oxford in 1649 and John Owen (the "Atlas of Independency" and a close associate of Cromwell) was named vice-chancellor of Oxford in 1652.

The more radical elements in the English revolution (including such sects as the Levellers, Ranters, Diggers, and the like) were not content with such measures, and questioned the very desirability of a state church and universities dedicated to it.[14] The radicals opposed the system of tithes that supported the state church and insisted that autonomous, freely assembled local congregations could and should support their chosen ministers. These "mechanick preachers" need not be university-educated men who had mastered the obscure subtleties of "school divinity," but instead plain men called to preach by the direct inspiration of the Holy Spirit. In the view of the radicals, the reform of the universities involved much more than simply ensuring that the properly reformed theology was taught or ejecting Royalists and Anglicans from their university livings. They campaigned rather for reorienting the university away from its historic mission of training preachers and bringing it to the service of a broader population by promoting "useful learning" in the arts and sciences.

2.3.1 *John Webster's* Academiarum Examen

John Webster (1610–82) was a radical with close ties to Cromwell's army, where he served as both surgeon and chaplain. His 1654 treatise *Academiarum Examen* set forth the radical program for the universities, calling for nothing less than expunging theology from the university curriculum and replacing the doctrines of the schools with a new learning that combined Baconian principles with occult and alchemical researches into "natural magic." According to Webster, the universities are burdened by their heritage of Scholasticism, so much so that

14. See Hill 1975 for a study of the radical program for the universities. Debus 1970 deals with the critique of universities and its relationship to the scientific developments of the period. For an overview of the different radical Puritan sects see McGregor and Reay 1984 and Mullett 1994.

the opinion of Aristotle holds sway in all matters touching on natural philosophy. In language that could have come straight out of Hobbes's *Leviathan,* Webster laments that "the *Philosophy* which the *Schools* use and teach, being meerly *Aristotelical,* [has been] imbraced and cryed up more than all others [so that] he should be accounted the Prince of *Philosophers,* the Master-piece of Nature, the Secretary of the Universe, and such an one beyond whose knowledge there is no progression" (1654, 52). This alleged monopoly of Aristotelianism has prevented the universities from applying themselves to the study of nature and has reduced natural philosophy to a literary exercise in the interpretation of Aristotle:

> This School Philosophy is altogether void of true, and infallible demonstration, observation, and experiment, the only certain means, and instruments to discover, and anatomize natures occult and central operations; which are found out by laborious tryals, manual operations, assiduous observations, and the like, and not by poring continually upon a few paper Idols, and unexperienced Authors. (1654, 68)

The remedy for this ill is to banish Aristotelian natural philosophy from the universities and institute a program of study in "Pyrotechny or Chymistry," grounded particularly in the hermetic-magical tradition of Paracelsus, Jan Baptist van Helmont, and Robert Fludd. The anticipated result of this program is "that the Schools therefore would leave their idle, and fruitless speculations, and not be too proud to put their hands to the coals and furnace, where they might find ocular experiments to confute their fopperies, and produce effects that wou'd be beneficial to all posterities" (1654, 71).

Webster finds fault with the theology of the schools as well. The influence of Scholastic teachings in theology has corrupted the simple, pure message of the Gospel and produced "a confused *Chaos,* of needless frivolous, fruitless, triviall, vain, curious, impertinent, knotty, ungodly, irreligious, thorny, and hel-hatch't disputes, altercations, doubts, questions and endless janglings, multiplied and spawned forth even to monstrosity and nauseousness" (1654, 15). In fact, Webster finds the enterprise of school divinity opposed to true Christian principles precisely because it keeps the Gospel and its interpretation in the hands of vain and ambitious doctors of divinity who have monopolized the spiritual life of England. In some of the seventeenth century's more purple prose, Webster complains that

> [f]rom this putrid and muddy fountain [of school divinity] doth arise all those hellish and dark foggs and vapours that like locusts crawling from this bottomlesse pit have overspread the face of the whole earth, filling men with pride, insolency, and self-confidence, to aver and maintain that none are fit to speak, and preach the spiritual & deep things of God, but such as are indued with this *Scholastick*, & mans *idol-made learning*, and so become fighters against God, and his truth, and persecutors of all those that speak from the principle of that wisedome, *that is from above, and is pure and peaceable:* not confessing the nothingness of creaturely wisedom, but magnifying, and boasting in that which is *earthly, sensual,* and *devilish.* (1654, 12)

The new university envisaged by Webster would not train preachers, because "the teaching of spiritual and Gospel knowledge is onely and peculiarly appropriated and attributed unto the Spirit of God" (1654, 4). Instead, Webster's properly reformed university would eschew theology altogether and teach "onely humane science," which is now regarded as too low and common for the attention of educated men. The result will be a university that takes as its main function the expansion of knowledge of nature and the implementation of this knowledge for practical improvements in the lives of the generality of mankind.

2.3.2 *William Dell*

William Dell, whom the Parliamentarians had appointed master of Gonville and Caius College in Cambridge, was every bit as radical as Webster in his proposed reforms of the university and the abolition of tithes. Indeed, his writings against university learning have a strong "levelling" strain that is ironic in a man who held a position of considerable power within the university.[15] Like Webster, Dell thought that Oxford and Cambridge paid insufficient attention to the practical consequences of learning, and he proposed that new universities be set up in every major city to instruct the common folk in the arts and sciences. Once the monopoly of Oxford and Cambridge was broken, the benefits of education could be extended to the more deserving lower orders, while righteous men called to preach the Gospel and freely chosen by their congregations could replace the caste of state-supported

15. See Hill 1975, 136–42, and Webster 1973 for accounts of Dell's radical program for the universities. Walker 1970 is a biography that downplays Dell's radicalism and treats his writings on the universities as an unfortunate aberration.

ministers. As he explains in his sermon on "The Right Reformation of Learning, Schools and Universities,"

> [I]f human learning be so necessary to the knowledge and teaching of the scriptures, as the universities pretend, they surely are without love to their bretheren, who would have these studies thus confined to [Cambridge and Oxford], and do swear men to read and teach them no where else: certainly it is most manifest, that these men love their own private gain, more than the common good of the people. (Dell 1816, 588–89)

Dell's educational remedy proceeds from the principle that "it would be more suitable, and more advantageous to the good of all the people, to have universities or colleges, one at least, in every great town or city in the nation" (Dell 1816, 589).

Dell also agreed with Webster in holding that true ministers of the gospel are not produced by the Scholastically tainted theological training of the universities, and he was as harsh in his condemnation of the errors produced by such theological training. In his "Testimony from the Word against Divinity Degrees in the University," Dell complains that

> the university, through power received from antichrist, [gives] men, chiefly for money, divinity degrees; and through those degrees, it gives authority and privilege to batchelors in divinity to expound part of the scriptures, and to doctors to expound and profess all the scriptures; and they that gain these degrees to themselves, are (as there is good reason) the great men in account with the university, and also with the carnal people of antichrist, how destitute soever they be of the faith and Spirit of the Gospel. (Dell 1816, 564)

The universities "are those antichristian soldiers, who put a reed into Christ's right hand instead of a sceptre: and their reed is philosophy, that vain deceit" (Dell 1816, 500). Thus, by expunging the writings of heathen philosophers (i.e., Aristotle) from the university, and eradicating the institution of divinity degrees, Dell hopes to reform the university in essentially the same manner as Webster.

The program of the radicals was anathema to Hobbes in many respects. In particular, the radicals' insistence upon immediate inspiration (rather than school divinity) as the only true path to the interpretation of Scripture is precisely the kind of emphasis upon "private judgement" that he denounced. Nevertheless, Hobbes and the radicals

shared a diagnosis of what was wrong with the universities and a general prescription of how it should be fixed. Specifically, they agreed that the universities were so heavily under the influence of Scholastic teachings that they could not be trusted to reform themselves, and they distrusted the capacity of cynical and ambitious doctors of divinity to cooperate with any move toward reform. Rather than rely upon a relatively autonomous university to cast off the pernicious influence of Aristotle, Hobbes and the radicals insisted that state power be brought to bear in forcing the universities to change their ways.

2.3.3 *Oxford's Response*

These criticisms of the universities did not pass without notice. Those in control of the universities were threatened by the calls for radical reform, and they had evident reason to fear the consequences their institutions might face. Vice-chancellor Owen, addressing Cromwell in December of 1654, reports that "we live among men ignorant of our ways, making our position slippery and uncertain. Then it came about that by listening carefully we heard a calumny. Once the pride and delight of the nation, we were soon almost a laughing stock" (Owen 1970, 79). Concern for the future of the universities was heightened in 1654 when the Barebones Parliament entertained a proposal for the outright abolition of the universities (Malcolm 1988, 54). Ward took the offensive against the universities' critics, and in 1654 published a detailed rebuttal to Webster under the title *Vindiciae Academiarum,* which included appendices replying to Hobbes and Dell.[16] There is a quite remarkable irony in the fact that Hobbes was explicitly linked to Webster and Dell. On matters of theology, church government, and politics, Hobbes was diametrically opposed to the views of these reformers; Ward was sensitive to the irony and remarked "how scornfully he will take it to be ranked with a Friar and an Enthusiast" ([Ward] 1654, 51).

The appendix directed at Hobbes was the first shot in the war that would last the next quarter century. In fact, it is virtually certain that Hobbes was led into his long-running mathematical controversy with

16. Although it was an anonymous work of Ward's (with an anonymous preface by Wilkins), we can identify the authors of the different parts of the *Vindiciae Academiarum.* As the *Dictionary of National Biography* entry for Webster notes, "The book *[Academiarum Examen]* was answered by Seth Ward, bishop of Salisbury, under the signature of H. D., the final letters of both his names, with a prefatory epistle by John Wilkins, bishop of Chester, also signed with final letters, N. S., and which has in consequence been assigned to Nathaniel Stephens."

Wallis by the goading of Ward and Wilkins. In the prefatory epistle to *Vindiciae Academiarum* Wilkins portrays Hobbes as "a person of good ability and solid parts, but otherwise highly magisteriall, and one that will be very angry with all that do not presently submit to his dictates, And for advancing the reputation of his own skill, cares not what unworthy reflexions he casts on others" (1654, 6–7). These charges are accompanied by insinuations of plagiarism from the manuscripts of Walter Warner[17] and a challenge to Hobbes to publish his solutions to the great geometrical problems such as the quadrature of the circle.[18] Ward betrays a familiarity with Hobbes's geometrical pretensions, presumably gathered through contact with friends and associates of Hobbes, and he strongly hints that there are those in Oxford who are prepared to debunk his claims to mathematical glory:

> Geometry hath now so much place in the Universities, that when Mr *Hobbs* shall have published his Philosophicall and Geometricall Pieces, I assure my selfe, I am able to find a great number in the University, who will understand as much or more of them then he desires they should, indeed too much to keep up in them that Admiration of him which only will content him. (1654, 58)

Ward's familiarity with Hobbes's scientific work makes it highly likely that he had knowledge of the state of his mathematical undertakings, and the unavoidable suspicion is that he hoped to provoke him into publishing an inept quadrature of the circle that would diminish Hobbes's mathematical reputation. In the event, the Monster of Malmesbury took the bait and hastily added chapter 20 to *De Corpore*, in

17. "It were not amisse, if he were made acquainted, that for all his slighting of the Universityes, there are here many men, who have been very well versed in those notions and Principles which he would be counted the inventer of, and that before his workes were published. And though he for his part may think it below him to acknowledge himselfe beholding to Mr *Warners* Manuscripts, yet those amongst us who have seen and perused them must for many things give him the honour of precedency before Mr *Hobbs*" (1654, 7). Prins (1993) has argued that this charge is without merit.

18. "I have heard that M. *Hobbs* hath given out, that he hath found the solution of some Problemes, amounting to no lesse then the Quadrature of the Circle, when we shall be made happy with the sight of those his labours, I shall fall in with those that speake loudest in his praise, in the meane time I cannot dissemble my feare, that his Geometricall designe (as to those high pieces) may prove answerable to a late Opticall designe of his, of casting Conicall glasses in a mould, then which there could not be any thing attempted lesse becoming such a man, as he doth apprehend himselfe to be" ([Ward] 1654, 57).

which he set forth a circle quadrature whose ineptitude probably surpassed Ward's fondest wishes.[19]

Meanwhile, Wallis was lying in wait.[20] He obtained the sheets of *De Corpore* from the printer as it was being printed, and prepared diligently to attack the mathematics in Hobbes's treatise as soon as it appeared in public. *De Corpore* was published in April of 1655, and Wallis's *Elenchus* appeared in August of the same year. Even by Wallis's relatively prolific standards, this is rather remarkable: he seems to have completed over 130 pages of Latin mathematical diatribe in a matter of months, all of it aimed at refuting every mathematical claim in *De Corpore* while interspersing other comments on the Hobbesian enterprise. Ward was busy as well, although he did not match the pace set by Wallis. By evident prior arrangement with Wallis, Ward attacked Hobbes's metaphysics, epistemology, theology, and political theory in his *In Thomae Hobbii Philosophiam Exercitatio Epistolica* (1656)—over 350 pages of anti-Hobbesian polemic.

The evidence assembled to this point should make it appear reasonable that Wallis's attack on Hobbes's geometric work has a significant connection with the contemporary debates over the universities. And, in fact, when we consider Wallis's own pronouncements on the matter, it becomes quite clear that his intention was not simply to expose some flaws in Hobbes's putative circle quadratures, but rather to con-

19. It is difficult to be certain whether Hobbes inserted the quadrature in chapter 20 as a direct result of the challenges issued by Ward. The chapter seems out of place, and reads rather like a late insertion. Hobbes hastily went through at least three versions of the failed quadrature, abandoning two previous efforts after friends had pointed out errors. Wallis (whose detective skills were honed by cryptographic efforts on behalf of the Parliamentarians in the Civil War) noted that chapter 20 had been printed, cut, and revised twice, basing his conclusion on an examination of the gatherings and the figures in the printed version of *De Corpore* (*Elenchus* 2–3, 95–96). I will investigate the background to the circle quadrature in *De Corpore* more closely in chapter 3.

20. Hobbes had the extraordinarily bad timing to publish *De Corpore* at the time when Wallis was preparing his *Arithmetica Infinitorum* for publication and was deeply involved with questions of circle quadrature. One of the principal results of *Arithmetica Infinitorum* is Wallis's method of representing π as an infinite product, and his researches in this area were ideal preparation for a frontal assault on Hobbes's mathematics. Wallis himself acknowledged as much: "having set aside other things, I gathered together these geometric pieces of yours, eager to see those things which mathematicians have sought in vain through every past century, and which you hope at last to have found. And I did this in principal part because this office [of Savilian Professor] did seem to demand it; and in part because I had been engaged in the same arena (nor, plainly, did I work in vain); and indeed I had brought the matter so far that four years ago I discovered the proposition on circle quadrature which I have recently made public" (*Elenchus* 2).

vince the reading public that the entire Hobbesian philosophy was ill founded. Recall that Hobbes had presented *De Corpore* as the first and foundational part of his philosophical enterprise, and he held that his doctrines were sufficiently well grounded that they would enable the solution of all problems. In Hobbes's scheme, the recognition that only body is real leads naturally to the conclusion that mathematics must be a generalized science of body, and a reformation of mathematics can then be effected by recasting the traditional principles of geometry in a more clearly materialistic form. Because the fundamental nature of body is clearly apprehended, this program (so Hobbes thinks) will make short work of the most difficult geometric problems, and the truth and power of the Hobbesian philosophy will be made manifest by its success in the solution of outstanding problems. Wallis intends to turn the tables on Hobbes, taking the demolition of the geometric work as a means to show that the grand program of its master is mistaken.

In a letter to Huygens in January of 1659 Wallis made his motivations for attacking Hobbes explicit, particularly as they relate to the universities. There he wrote:

> But regarding the very harsh diatribe against Hobbes, the necessity of the case and not my manners led to it. For you see, as I believe, from other of my writings how peacefully I can differ with others and bear those with whom I differ. But this was provoked by our Leviathan (as can be easily gathered from his other writings, principally those in English), when he attacks with all his might and destroys our universities (and not only ours, but all, both old and new), and especially the clergy and all institutions and all religion. As if the Christian world knew nothing sound or nothing that was not ridiculous in philosophy or religion; and as if it has not understood religion because it does not understand philosophy, nor philosophy because it does not understand mathematics. And so it seemed necessary that now some mathematician, proceeding in the opposite direction, should show how little he understands this mathematics (from which he takes his courage). Nor should we be deterred from this by his arrogance, which we know will vomit poison and filth against us. (Wallis to Huygens, 1/11 January 1659; *HOC* 2:296–97)

Wallis repeated a similar sentiment at the beginning of the edition of his *Opera Mathematica,* first published in 1693: "Certain works writ-

ten long ago against Thomas Hobbes (pseudo-geometer) will not be found here, for I would not want to seem to triumph over one now dead. Nevertheless, as things then stood, it was something that had to be done, when he had set himself up in the guise of a great geometer and dared to offer false suggestions to our unsuspecting youth in matters of religion" (*OM* 1:sig. a4v). The structure of Wallis's *Elenchus* reflects his concern with a thorough "root and branch" eradication of Hobbes's philosophy. Rather than confining himself to showing the shortcomings in Hobbes's attempted quadrature of the circle, Wallis argues that Hobbes's entire conception of mathematics is mistaken, and concludes that his opinions on other, weightier matters must be equally unreliable.[21]

In the dedicatory epistle to John Owen in the *Elenchus*, Wallis explicitly raises the question "why, leaving aside theological and other philosophical matters, I should undertake to refute his geometry, when he has made other far more dangerous errors?" His response is revealing, for he expresses a strong concern to defend Oxford against the "sheer calumny which he is ever trying to persuade people, that there are no mathematicians, or at least not in the university." But he is equally concerned

> that these things should be speedily opposed, not, to be sure, because I think that geometry should be protected lest it suffer harm (indeed this of all disciplines has least to fear from logical fallacies), but so that when this balloon has been burst, that man, so full of airy talk, might be quite deflated and that others, less skilled in geometry, may know that there is nothing more to be feared from this Leviathan on this account since its armor (in which he had the greatest confidence) is easily pierced: and also so that outsiders (if they saw him maintain such things unchecked) might not think all men who practice geometry here are like him. (*Elenchus* epistle, sig. A3v–A3r)

It is important to notice that this campaign against Hobbes makes sense only if Wallis has reason to think that Hobbes's views were taken seriously. The project of determining Hobbes's influence is a complex

21. Thus, in justifying his attention to Hobbes's mathematics, he declares, "But I do not [confine my attention to the geometric aspects of *De Corpore*] because I think that the remainder is better, for whoever stumbles so horribly in geometry, where demonstrative proofs have a place, can hardly be thought to walk more securely in other matters, where conjectures will often have to be made" (*Elenchus* 4).

and difficult undertaking, and one best left for another day, but it is clear that he must have been a force to be reckoned with.[22]

In the course of their dispute, Hobbes and Wallis debated issues that went far beyond questions of circle quadrature or the nature of mathematics; fine points of Latin grammar (such as the proper use of the ablative case), problems of Greek etymology, and questions of ecclesiastical government were all brought into play. Indeed, their battle assumed a life of its own and continued long after the Restoration had put to rest the debate over university reform from the 1650s. Disputes over the nature and function of the universities form the background for the initial conflict between Hobbes and Wallis, but they were irrelevant to its continuation through the 1660s and beyond. As might also be expected, strategic alliances were formed and disrupted in the course of the battle. The most salient example of this phenomenon concerns Wallis and Owen. They began by making common cause in the defense of the universities, with Wallis dedicating his *Elenchus* to Owen; but they later quarreled when Wallis attacked Independency by publishing a defense of Presbyterian principles.[23]

Having concluded our study of the social and ideological background to the dispute, we must now look at Hobbes's philosophy of mathematics, since an important element in the controversy is the clash between Hobbes's thoroughly materialistic interpretation of mathematics and Wallis's more traditional account of the subject.

22. Malcolm expresses a similar view of Ward and Wilkins' joint attack on Hobbes, when he observes, "It was precisely because Hobbes still appeared in 1654 as an authoritative speaker on behalf of the new science that Wilkins and Ward took such trouble to attack him" (Malcolm 1988, 54). See Rogers 1988 for an interesting and important study of Hobbes's influence and reactions to the Hobbesian philosophy.

23. The particular thesis that bothered Owen was Wallis's negative response to the question "Whether the power of the evangelical minister extends wholly to the members of one particular church?" in a public disputation in 1654 and later published as part of a collection of theological works (Wallis 1657a). These matters will be investigated in chapter 7.

CHAPTER THREE

De Corpore and the Mathematics of Materialism

I am the first that hath made the grounds of Geometry firm and coherent. Whether I have added anything to the Edifice or not, I leave to be judged by the Readers.
—Hobbes, *Six Lessons*

The publication in 1655 of Hobbes's treatise *De Corpore* was the advent of the first part of his complete three-part system of the elements of philosophy. Although first in Hobbes's envisioned logical order of exposition, *De Corpore* was not the first part of the system to be published, appearing as it did four years after *De Cive*—the third part of the system. Taken together, Hobbes's three works *De Corpore, De Homine,* and *De Cive* were intended as an exposition of all the philosophy worth knowing. The structure of the three works reflects Hobbes's conception of the structure of knowledge: beginning with the nature of body, they proceed to the nature of man (i.e., an animated, rational body), and thence to the nature of the artificial body of the commonwealth, formed by the covenants that bind men together. The result is that *De Corpore*—a dissertation on the nature of body—is the foundational work in the Hobbesian system, and its status as a foundation derives from the fact that Hobbes held to a strict materialism in which only body is truly real and all else is to be explained as the consequence of the motion and impact of bodies.

Even a casual perusal of this work reveals that Hobbes's exposition of the elements of philosophy led him into territory far removed from anything we today regard as strictly philosophical. In particular, chapters 12–20 deal with mathematical material that seems out of place in a treatise on the nature of body. These chapters contain an extended discussion of the nature and methods of mathematics and include (among other claimed geometric results) a putative quadrature of the circle. Despite appearances, the presence of this mathematical material

in *De Corpore* is by no means an oversight or an anomaly. Rather, Hobbes's excursion into mathematics reflects his conception of the unity of all knowledge, as well as his confidence that a thoroughly materialistic treatment of mathematics will provide the solution to all mathematical problems. Philosophy, as Hobbes declares in the opening chapter of *De Corpore* "*is knowledge of the effects or phenomena acquired by right reasoning from their known causes or generations, and conversely of such generations as may be from known effects*" (*DCo* 1.1.2; *OL* 1:3). This broad conception of the philosophical enterprise counts as philosophical any body of knowledge grounded in the consideration of causes, and more specifically, mechanical causes. A key part of Hobbes's program is to set forth the true (mechanical) causes of mathematical objects and thereby make mathematics a branch of philosophy.

The guiding principle in this philosophico-mathematical program is that mathematics is a science whose object is the principal affections of body. We have seen that mathematics was traditionally characterized as the science of quantity, where quantities (whether discrete or continuous) are anything capable of being measured or counted. Hobbes can accept this definition of mathematics, but adds the condition that the three dimensions of body (length, breadth, and depth) are the only genuine subjects of quantity since "there is no Subject of Quantity, or of Equality, or of any other accident but Body" (*SL* 2; *EW* 7:226–27). This condition follows quite naturally from his thoroughgoing materialism: because he holds that only body is real, Hobbes must also hold that mathematical quantities (numerical as well as geometrical) are ultimately grounded in the nature of body. The philosophy of mathematics set forth in *De Corpore* (particularly in chapters 12–14) is thus Hobbes's attempt to show how all of mathematics can be interpreted as a science of body.

In adopting this approach Hobbes departs quite radically from accepted doctrines in the philosophy of mathematics, and particularly the traditional understanding of mathematical objects. The standard Scholastic classification of the mathematical sciences relied upon a distinction between pure and "mixed" mathematics in which the pure science studies quantities abstracted from physical or material circumstances. Mixed mathematics deals with actual physical objects that embody or instantiate (perhaps imperfectly) the characteristics of the objects of pure mathematics. This view has Aristotelian origins but was widely accepted in Hobbes's day, not least because it allows mathematics to be integrated into the traditional conception of science in which

mathematics is a mean between physics (which must remain at the level of material things) and metaphysics (whose objects are such immaterial things as God, the soul, or being in general). Clavius phrases the issue this way in his commentary on Euclid's *Elements:*

> Because the mathematical sciences treat of things that are considered apart from all sensible matter, although they are themselves immersed in matter, this is the principal reason that they occupy a middle position between the metaphysical and natural sciences. If we consider [the sciences each according to its] subject, as Proclus correctly pronounced, the subject of metaphysics is indeed separated from all matter of any kind. But the subject of physics is always conjoined with some kind of matter. Whence, as the subject of the mathematical disciplines is considered apart from all matter, it is clear that it constitutes a mean between the other two. (Clavius 1612, 1:5)

On this view, the Euclidean line (defined as "length without breadth") is an abstraction in which length is mentally separated from breadth. Although there are no breadthless lengths to be found in nature, pure mathematics is not constrained to deal with things actually existent, and its theorems are true independent of the structure or contents of the physical world. A mixed science such as astronomy then treats physical objects as approximations to the abstractions of pure mathematics: a light ray can be considered as an abstract line, for example, or a planet can be treated as a point with respect to its orbit. According to Barrow's version of the distinction: "The pure or abstract parts of mathematics contemplate the nature and proper affections of magnitude and number; but the mixed or concrete parts consider the same as applied to certain bodies and particular subjects, combined with motive force and other physical accidents. Whence it is that Aristotle calls them φυσικωτέρας (*Physics*, Book 2) and αἰσθήτικας (*Posterior Analytics* 1.33) and others are accustomed to call them physico-mathematical" (*LM* 1, 1:31).[1]

1. The Aristotelian passages Barrow has in mind are the following. "Similar evidence is supplied by the more natural of the branches of mathematics, such as optics, harmonics, and astronomy. These are in a way the converse of geometry. While geometry investigates natural lines but not qua natural, optics investigates mathematical lines, but qua natural, not qua mathematical" (*Physics* 2.2; 194a7–11). "The reason why differs from the fact in another fashion, when each is considered by means of a different science. And such are those that are related to each other in such a way that the one is under the other, e.g., optics to geometry, and mechanics to solid geometry, and harmonics to arithmetic, and star-gazing to astronomy" (*Posterior Analytics* 1.13; 78b34–79a2).

In taking body as the fundamental object of mathematics Hobbes rejects this widely held "abstractionist" philosophy of mathematics and sets himself the task of developing an alternative to the received view. Chapters 12–14 of *De Corpore* are his attempt to construct an account of quantities, ratios, and figures consistent with his materialism, while chapters 15–19 are intended to exploit this account in the investigation of motion and the determination of ratios between magnitudes. Chapter 20 contains several ill-fated attempts to apply this conception of mathematics to the problem of squaring the circle.[2] In the course of these investigations, Hobbes covers ground familiar from the history of the philosophy of mathematics, but his aim is always to impose a new set of definitions on traditional subject matter. Although my exposition here concentrates on Hobbes's views as set forth in *De Corpore,* the complexity of the issues will occasionally make it necessary to consider some of his later writings, as well as those of other authors.

3.1 HOBBES ON THE NATURE OF QUANTITY

The foundation of the Hobbesian program for mathematics is set forth in chapter 8 of *De Corpore*, which bears the title "Of Body and Accident." In his discussion of body and its principal accidents, Hobbes introduces several key definitions that he later exploits in his more detailed discussion of the nature of mathematics in chapters 12–14. Article 4 of chapter 8 casually defines the key term *magnitude* with the remark that "the extension of a body is the same as its magnitude, or that which some call 'real space'" (*DCo* 2.8.4; *OL* 1:93).[3] This identification of magnitude with space is pursued further in article 12 of the same chapter, when Hobbes introduces his definitions of three fundamental geometric objects—point, line, and surface. As he declares:

> If the magnitude of a body which is moved (although it must always have some) is considered to be none *[nulla]*, the path by which it travels is called a *line* or one simple dimension, and the

2. There are significant differences between the scheme of printed chapters and those that survive in manuscript, which indicates that Hobbes's intentions underwent some significant changes in the course of writing *De Corpore*. I will have more to say about this when we consider the background to the printing of the circle quadratures in the twentieth chapter of *De Corpore*.

3. On Hobbes's conception of space, and particularly the doctrine of "real space," see Leijenhorst 1998, chap. 3, pt. 1.

> space it travels along a *length*, and the body itself is called a *point*. This is the sense in which the earth is usually called a point and the path of its annual revolution the ecliptic line. But if a body that is moved is considered now as long and it is supposed to move so that each of its parts is understood to make a line, the path of each and every part of the body is a *breadth* and the space produced is called a *surface*, consisting of two dimensions, *breadth* and *length*, of which the whole of one is applied to the single parts of the other. (*DCo* 2.8.12; *OL* 1:98–99)

These definitions extend naturally to cover the generation of solids, and Hobbes connects the nature of solids to the three-dimensionality of space in the remainder of article 12:

> Again, if a body is now considered as having a surface, and is understood to be so moved that each of its single parts makes lines, then the path of all these parts is called thickness or depth, and the space made is called a solid, consisting of three dimensions any two of which are entirely applied to all the parts of the third.
>
> But if a body is further considered as a solid, it cannot happen that each of its parts describes single lines. For however it may be moved, the path of the following part will coincide with the preceding part, and the same solid will be made that would have been made by the first surface itself. And so there can be no other dimension of body, as it is a body, beyond the three mentioned. Although, as will be said later, velocity, which is motion through a length, applied to all parts of a solid, makes a magnitude of motion consisting of four dimensions, just as the worth *[bonitas]* of gold computed in all of its parts makes its price *[pretium]*. (*DCo* 2.8.12; *OL* 1:99)

The significance of these definitions for Hobbes's philosophy of mathematics can scarcely be overstated, and we will see their consequences in nearly all of his mathematical writings. For present purposes it suffices to stress the connection between these definitions and Hobbes's strict materialism. Because his ontology recognizes only body as real, and because he holds that it is only through motion of bodies that anything can be brought about, Hobbes's program in the philosophy of mathematics must found all of geometry upon the principles of matter and motion. That is to say, mathematical objects must be interpreted as bodies or as things produced by the motion of bodies.

As I have indicated, Hobbes's program is a clear departure from the traditional conception of geometric magnitudes. Euclid, for example, defines a point as "that which has no parts," while a line is "breadthless length," and a surface is "that which has length and breadth only" (*Elements* 1, defs. 1, 2, 5). Philosophers and philosophically minded mathematicians after Euclid accepted these definitions and typically interpreted them as expressing the fundamentally immaterial nature of geometric objects, since no physical bodies can satisfy the definitions. Proclus held that the Euclidean definitions articulated the fundamental properties of forms separable from matter, so that geometric definitions cannot accurately be interpreted as true of material things. He explains that

> in the forms separable from matter the ideas of the boundaries exist in themselves and not in the things bounded, and it is because they remain precisely what they are that they become agents for bringing to existence the entities dependent upon them. . . . Matter muddies their precision; the idea of the plane gives the plane depth, that of the line blurs its one-dimensional nature and becomes generally divisible, and the idea of the point ends by becoming bodily in character and extensible together with the thing that it bounds. For all ideas when they flow into matter . . . are filled with their substrates: they forsake their native simplicity for alien combinations and extensions. (Proclus 1970, 87)

Similarly, Clavius's comment on the Euclidean definition of a point contains the remark that "[n]o example of this can be found in material things, unless you mean that the extremity of the sharpest needle expresses some similitude to a point; which nevertheless is wholly untrue, since this extremity can be divided and cut to infinity, but a point must be supposed altogether indivisible *[individuum prorsus]*" (Clavius 1612, 1:13).

Such remarks as these show the prevalence of the idea that (pure) mathematics is not at all concerned with the nature of physical body and underscore just how radically Hobbes's program departs from the traditional understanding of geometry and its object. The extent of his departure from the received view can be gauged by Wallis's apoplectic comments on the definition of *point* in *De Corpore:*

> Who ever, before you, defined a point to be a body? Who ever seriously asserted that mathematical points have any magnitude?

> If it is a body, if it has size, then one point added to another will make a greater. For you will not, I think, deny this of bodies, therefore neither will you deny it of points, if indeed a point is a body. But if the addition of one point adds nothing, then neither will the addition of two, three, a hundred, a thousand, or even an infinity of points. Indeed, you elsewhere argue in this way about impetus, so why does the same not hold of magnitude? Perhaps you will say that the magnitude itself, so far as there is any, is nevertheless not considered. Fine. But I now ask why is it still considered to be a body? If it is not, then to what purpose is this mention of body in the definition, or why cannot this corporeity also be excluded, and also this magnitude? Why do you not then define it thus, *A point is a body which is not considered to be a body, and a magnitude [magnum] which is not considered to be a magnitude?* (*Elenchus* 6–7)

Hobbes retorts that the Euclidean definition of a point is ambiguous and can lead to absurdity. As he notes, we can interpret the talk of a point having no parts to mean either that a point is indivisible or undivided. He maintains that in the first sense a point is simply nothing:

> That which is indivisible is no Quantity; and if a point be not Quantity, seeing it is neither substance nor Quality, it is nothing. And if *Euclide* had meant it so in his definition, (as you pretend he did) he might have defined it more briefly, (but ridiculously) thus, *a Point is nothing*. (*SL* 1; *EW* 7:201)

Defining a point as something undivided but capable of further division will indeed make it a quantity, but such a definition fails to express the essence of a point. For Hobbes, the essence of a point is that its magnitude is not considered in a demonstration, and his alternative definition captures this essential feature of the point and places all of geometry on a firm foundation.

In defense of his definition of *point,* Hobbes even tries to argue that it is consistent with the opinions of Euclid and Proclus, since both classical authors admit that a point cannot be considered in the course of a geometric argument.[4] However, in defining points as bodies, Hobbes

4. Proclus, in commenting upon Euclid's definition of a point remarks that "[i]n geometrical matter, then, the point alone is without parts, and in arithmetic the unit; and the definition of the point, though it may be imperfect from another point of view is perfect as far as the science before us is concerned. . . . It all but clearly says that 'what is without parts is a point for my purposes and a principle for me; and the simplest object is none other than this'" (Proclus 1970, 76–77). Hobbes takes this to mean that

is also committed to the thesis that points have extension (an unextended body being an utterly absurd concept, on Hobbes's or anyone else's principles). The notion of an extended point conflicts mightily with the traditional conception, and it also seems to demand that there be points of different sizes, since there could certainly be different sizes of divisible objects whose parts are not considered in a demonstration.[5]

Hobbes naturally finds the Euclidean definition of a line ("breadthless length") as flawed as the definition of a point, since he holds that a length without breadth is simply nothing. However, there is an alternative tradition according to which the line is produced by the motion of a point. Aristotle, for example, mentions such an understanding of points in *De Anima,* when he remarks that "since they say a moving line generates a surface and a moving point a line, the movements of the units must be lines" (*De Anima* 1.4; 409a4–5). Clavius reports a similar definition when he reports that "mathematicians also, in order to teach us the true understanding of the line imagine a point, as described in the above definition, to be moved from one place to another. Now since the point is absolutely indivisible, there will be left behind by this imaginary motion a certain path lacking in all breadth" (Clavius 1612, 1:13). Proclus rejects such an account as precisely the wrong way to think about geometric lines because it "appears to explain [the line] in terms of its generative cause and sets before us not line in general, but the material line" (Proclus 1970, 79).

Hobbes disagrees with Proclus and finds such a definition of the line congenial to his project. Clearly, the attempt to reduce geometry to motion and extension would be advanced by including the concept of motion in the definition of a line, and Hobbes is quite happy to include an explanation of the generative cause of the line. In this scheme, a line will indeed be the path of a moving point, but the point must be understood in Hobbes's terms:

"no Argument in any Geometricall demonstration should be taken from the Division, Quantity, or any part of a Point; which is as much as to say, a Point is that whose quantity is not drawn into the demonstration of any Geometricall conclusion; or (which is all one) whose Quantity is not considered" (*SL* 1; *EW* 7:201).

5. Hobbes embraces this consequence in *De Corpore* 3.15.2 when he says, "Although just as a point can be compared with a point, so a *conatus* can be compared with a *conatus,* and one can be found to be greater or less than another. For if the vertical points of two angles are compared to one another, they will be equal or unequal in the ratio of the angles, or if a right line should cut many circumferences of concentric circles, the points of section will be unequal in the ratio of these perimeters" (*DCo* 3.15.2; *OL* 1:178). I will have more to say about Hobbes's concept of *conatus* in section 4 of this chapter.

> But everyone knows that nothing except body can be moved, nor can motion be conceived of anything except body. And every body in motion traces a path with not only length, but also breadth. Therefore, the definition of a line should be as follows: a *line is the path traced by a moving body, whose quantity is not considered in a demonstration.* (*PRG* 2; *OL* 4:393)

Wallis objected to the Hobbesian definition of line because, among other things, the definition includes the concepts of motion and body, which Wallis regards as inessential to geometry.[6] In reply, Hobbes urges that not only do such classical authors as Euclid define geometric objects in terms of the motions by which they are generated, but that geometry can be made part of genuine philosophy only if its definitions contain the generation of the things it studies. Thus, geometric definitions must appeal to the motions by which geometric objects are created; or, as Hobbes puts it: "I say, to me, howsoever it may be to others, it was fit to define a Line by Motion. For the generation of a Line is the Motion that describes it. And having defined Philosophy in the beginning, to be the knowledge of the properties from the generation, it was fit to define it by its generation" (*SL* 2; *EW* 7:215). He further explains that by employing such definitions he has removed the grounds for skeptical doubts about geometry that had been raised by Sextus Empiricus in his treatise *Against the Mathematicians:*

> [W]here there is place for Demonstration, if the first Principles, that is to say, the Definitions contain not the Generation of the Subject; there can be nothing demonstrated as it ought to be. And this in the three first Definitions of *Euclide* sufficiently appeareth. For seeing he maketh not, nor could make any use of them in his Demonstrations, they ought not to be numbered among the Principles of Geometry. And *Sextus Empiricus* maketh use of them (misunderstood, yet so understood as the said Professors understand them) to the overthrow of that so much renouned Evidence of Geometry. In that part therefore of my Book where

6. Wallis asks, "Moreover, what mathematician ever expected such a definition of line or length? The things you affirm are indeed true, but they are not definitions. Neither are they reciprocal propositions. What need is there for the idea of motion, or of a body moved, in order that it be understood what a line is? For are there not lines in quiescent bodies, no less than moved ones? Why then this mention of motion? And will not the distance even between two quiescent bodies be a length, no less than the measure by transit? To what purpose then this mention of a body moved?" (*Elenchus* 6). These complaints will be investigated more fully in chapter 4.

> I treat of Geometry, I thought it necessary in my Definitions to express those Motions by which Lines, Superficies, Solids, and Figures were drawn and described; little expecting that any Professor of Geometry should finde fault therewith; but on the contrary supposing I might thereby not only avoid the Cavils of the Scepticks, but also demonstrate divers Propositions which on other Principles are indemonstrable. (*SL* epistle; *EW* 7:184–85)

Hobbes returns to the issue of skepticism in mathematics when he charges Wallis with employing definitions that place geometry in danger of succumbing to skeptical challenges, whereas his definitions can "redeem" geometry.[7] It is a disputed question among Hobbes scholars whether the philosopher from Malmesbury was deeply concerned with skepticism and the appropriate response to it, but from such comments as these it is evident that he thought his philosophy of mathematics had the notable advantage of forestalling skeptical attacks on geometry.[8]

The materialistic understanding of points, lines, and surfaces that Wallis finds so objectionable is just the beginning of the Hobbesian program for mathematics. Hobbes more fully expounds his philosophy of mathematics in chapter 12 of *De Corpore*, which bears the title "Of Quantity" and sets forth his leading ideas on the nature of quantities. A quantity, he declares, can be defined only as *"a determined dimension, or a dimension whose termini are known either by their place or by some other comparison"* (*DCo* 2.12.1; *OL* 1:124). The reason for defining quantities in terms of dimensions was touched on before and should be obvious enough: the three dimensions of length, breadth, and depth are fundamental affections common to all bodies, and thus make all body the subject of quantity. Defining quantities as *determined* dimensions requires that quantities be marked off or limited by definite boundaries, since quantities are always appropriate responses to a question about how great some body is. As Hobbes puts it:

7. Hobbes writes: "From this one and first Definition of *Euclide, a Point is that whereof there is no part*, understood by *Sextus Empiricus*, as you understand it, that is to say mis-understood, *Sextus Empiricus* hath utterly destroyed most of the rest, and Demonstrated, that in Geometry there is no Science, and by that means you have betrayed the most evident of the Sciences to the *Sceptiques*. But as I understand it for *that whereof no part is reckoned*, his Arguments have no force at all, and Geometry is redeemed" (*SL* 5; *EW* 7:317–18).

8. See Tuck 1988a and 1988b for a portrayal of Hobbes's system as a response to skepticism, and particularly to Descartes's "hyperbolic doubt." Sorell (1995) takes issue with this sort of interpretation and downplays its significance for Hobbes's philosophy.

> When, therefore, it is asked how much something is, for example how long is the road, it is not answered indefinitely "length," nor to one who asks how large is the field, indefinitely "surface," nor if someone asks how great is the bulk *[moles]*, indefinitely "solid." Rather, it is responded determinately, that the road is one hundred miles, the field is one hundred acres, or the bulk is one hundred cubic feet, or at least in some way that the magnitude of the thing asked after can be comprehended by certain limits in the mind. (*DCo* 2.12.1; *OL* 1:123)

The process of determining magnitudes depends in the first instance upon the motions by which dimensions are described, but it can also be performed by the "apposition" of previously determined quantities or the "section" of such determined quantities.[9] There are also two different ways to determine a quantity: either by immediate "exposition to sense" or by expressing it as a multiple of some common measure that itself is "recoverable to sense." The height of a wall, for instance, can be immediately sensed or grasped, or it can be expressed as a number of meters, where the meter itself is a common measure whose magnitude can be immediately sensed.[10]

This account of quantity gives pride of place to the continuous magnitudes of geometry, and it may at first sight seem to provide no means of capturing the classical distinction between discrete and continuous quantities. As I mentioned in chapter 1, classical authors divided mathematics into two sciences (geometry and arithmetic) on the basis of a

9. Hobbes writes:

> Lines, surfaces, and solids are first exposed by motion, in the way we said they are generated in chapter 8, in such a way that their traces remain; as when they are marked in matter, like a line on paper or carved in durable matter. The second way is by apposition, as when a line is added to a line, that is, a length is added to a length, or a breadth to a breadth, or a thickness to a thickness, which is to describe a line by points, a surface by lines, or a solid by surfaces, except that by points here is to be understood very short lines, and by surfaces very thin solids. The third way lines and surfaces can be expressed is by sections, namely a line may be made by cutting an exposed surface, and a surface may be made by cutting a solid. (*DCo* 2.12.3; *OL* 1:124–25)

10. Hobbes writes, "But quantity is determined in two ways; the one to sense, which may be done by a sensible object, as when a line, surface, or solid of a foot or a cubit is set before the eyes marked in some matter, which way of determining is called that of exposition, and the quantity thus known is called exposed quantity. In the other way quantity is determined by memory, which is done by comparison with an exposed quantity" (*DCo* 2.12.2; *OL* 1:124).

fundamental difference in their objects. Geometry takes as its object the continuous quantities such as lines, angles, and areas, while arithmetic deals with the discrete quantities or "number" that are collections of units. Hobbes admits such a distinction in *De Corpore*, but draws it in terms of the means by which the different kinds of quantities are determined or declared. Continuous quantities are determined by setting out their limits, and such determination ordinarily depends upon motion; numbers or discrete quantities, however, are exposed "by the exposition of points, or also by the numeral names *one, two, three*, etc. And these points should not be so contiguous to one another that they cannot be distinguished by marks *[nullis notis]*, but so positioned that they can be discerned. For it is by this that number is called *discrete* quantity, while all quantity which is determined by motion is called *continuous*" (*DCo* 2.12.5; *OL* 1:125). Elsewhere he argues that the discrete quantities of arithmetic can be generated from the uniform division of geometric magnitudes, and he thus shares Barrow's opinion that arithmetic is subordinate to geometry.[11]

The doctrine thus far presented fails to include many things that are ordinarily taken to be quantities—times, masses, velocities, densities, temperatures, etc. are all capable of measure and must therefore be integrated within the general framework of quantities in *De Corpore*. Hobbes's approach to this problem is to allow such further quantities to be "exposed" by the fundamental quantities, and particularly by points and lines in motion. Time, for example, is exposed "when not only some line is exposed, but also a movable thing *[mobile]* that is uniformly moved upon it, or is supposed to be so moved. For since time is the image of motion so far as in it earlier and later, that is succession, are considered, it is not sufficient for the exposition of time that a line is described, but also that there is in the mind an image of some movable thing passing over that line, and this with uniform motion, so that time can be divided and compounded as often as there is need" (*DCo* 2.12.4; *OL* 1:125).[12] Similar expositions of other quanti-

11. As Hobbes declares in the *Examinatio*, "[B]ecause any given continuous magnitude can be divided into any number of equal parts, with its ratio to any other magnitude remaining unchanged, it is manifest that arithmetic is contained in geometry" (*Examinatio* 3; *OL* 4:28).

12. It is worth noting that unless Hobbes takes the concept of uniform motion as undefined and unproblematic, this definition involves a troublesome circularity. To say that time is exposed by the uniform motion of a point across a line requires that the concept of uniform motion be understood; but if uniform motion is defined as motion covering equal distances in equal times, it is taken for granted that we have some way of exposing equal times, which was the very thing to be explained.

ties can then be made in the obvious way, by representing a quantity through lines and motions.[13] This approach thus provides a unified conception of quantity in which such apparently disparate quantities as time and temperature can all be reduced to or represented as aspects of body. There are other kinds of geometric quantities such as figures and angles whose Hobbesian treatment I will consider in section 3.3, but first it is necessary to examine Hobbes's doctrine of ratio and proportion.

3.2 RATIO AND PROPORTION IN *DE CORPORE*

The doctrine of ratios is an essential part of any geometric theory, and Hobbes was concerned to formulate a treatment of the subject that is of a piece with his general account of quantity. There were significant disputes in the seventeenth century over the proper interpretation of the theory of ratios, and it is necessary to touch on some of these contested issues to clarify both Hobbes's position and the doctrines it was intended to supersede. As we will see, Hobbes was familiar with the disputes regarding the nature of ratios and he offered his own theory as a solution to the difficulties encountered by other writers.

The starting point for an understanding of the doctrine of ratios and proportions is the classical theory developed in book 5 of Euclid's *Elements* as a general doctrine applicable to all geometric magnitudes. The principal definitions read as follows:

> 3. A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind.
> 4. Magnitudes are said to **have a ratio** to one another which are capable, when multiplied, of exceeding one another.
> 5. Magnitudes are said to be **in the same ratio,** the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of the latter equimultiples respectively taken in corresponding order.
> 6. Let magnitudes which have the same ratio be called **proportional.** (*Elements* 5, defs. 3–6)

13. The exposition of mass is a somewhat different case, since the mass of a body is exposed with the body itself. As Hobbes puts it, "But weight is exposed by any heavy body of any sort of matter, provided that it is always of the same heaviness [*gravitas*]" (*DCo* 2.12.7; *OL* 1:126).

It is important to observe that this understanding of ratios does not treat a ratio as a quotient of two numbers, but rather as a special kind of relation between two magnitudes. These magnitudes are conceived of as falling into species or kinds, and it is only within species that ratios can be constructed or quantities compared. For example, lines, angles, surfaces, and solids are different species of magnitudes, but the restrictions in definitions 3 and 4 prohibit the formation of ratios between lines and surfaces, since there is no number of lines that could ever exceed a surface, and thus there can be no "relation in respect of size" between heterogeneous magnitudes. The sixth definition allows comparison of ratios across species, so it makes sense to say that the ratio between two lines L_1 and L_2 is the same as the ratio between two spheres S_1 and S_2. The proportion $L_1{:}L_2 :: S_1{:}S_2$ would therefore be legitimate. Note, however, that the definition of equality of ratios (definition 5 above) does not assert that $A{:}B :: C{:}D$ whenever $A \times D = C \times B$, because the relevant magnitudes may be heterogeneous and incapable of being multiplied together. Instead, sameness of ratio is defined by the property of preserving order relations under arbitrary equimultiples, as expressed in the complicated fifth definition above.

The Euclidean theory of ratios explicitly prohibits the introduction of infinitesimal ratios (by definitions 4 and 5), but it does allow for incommensurable magnitudes to appear as terms in ratios. The exclusion of infinitesimal magnitudes is guaranteed by the conditions listed in definition 4, which restricts ratios to the consideration of finite differences between finite magnitudes. Incommensurable magnitudes may still appear as terms in a Euclidean ratio because the first definition allows ratios to be formed between any two magnitudes of the same kind. In particular, this allows any two line segments to be compared, so that the ratio between, say, the side and diagonal of a square can be formed in the classical theory, even though this ratio cannot be expressed as a ratio between two integers.

The classical doctrine (which we can call the "relational" theory of ratios) was not universally accepted in the seventeenth century, for some authors preferred an understanding of ratios that we can call the "numerical" theory.[14] The differences between these can be illustrated by asking whether ratios are quantities. According to the relational theory, the answer is no: ratios are not themselves magnitudes, but

14. I take this terminology from Sylla 1984. The numerical theory did not originate in the seventeenth century, but can be traced back through the Middle Ages to late antiquity. Nevertheless, the clash between the relational and numerical theories was most evident in the seventeenth century.

rather relations that hold between pairs of magnitudes. Just as it is absurd to hold that the numerical relation "less than" is a number, the relational theory of ratios denies that there is any sense to the assertion that ratios are quantities.

In contrast, the numerical theory of ratios assimilates ratios into the general domain of magnitudes. In the numerical presentation, every ratio has a size or "denomination," and two ratios are the same if and only if they have the same denomination. By assigning denominations to ratios, the numerical theory avoids the elaborate Euclidean definition of equality of ratios in terms of arbitrary equimultiples, and this is one of its main attractions. Nevertheless, the theory is not without its difficulties. From the standpoint of the numerical theory it is natural to assume that the ratio $A{:}B$ is the same as the ratio $C{:}D$ just in case $A \times D = B \times C$. As we have seen, such a criterion makes no sense under the relational theory because the magnitudes A and D may be of different species and therefore incapable of being compared directly with one another. Thus, a fully developed numerical theory of ratios must require that all ratios be homogeneous and capable of mutual comparison. Further, it is natural to see the denominations as answering to quotients, since a ratio is taken to be a magnitude that arises from the division of the antecedent by the consequent. Yet in identifying ratios with quotients the numerical theory requires that the quotient of two incommensurable magnitudes be defined. The quotient is understood classically as arising from the division of integers; deriving from the Latin *quoties* or "how many," the quotient is taken to indicate how many common units from the denominator are contained in the numerator, and in effect quotients can simply be identified with rational numbers. A full vindication of the numerical theory therefore requires expanding the classical understanding of quotients to include quotients of incommensurable magnitudes. As might be expected, these debates between the proponents of the numerical and relational theories of ratios were a source of considerable interest to Hobbes.

The conflict between these competing theories of ratios led to some significant debates in the seventeenth century, although not all authors explicitly raised the issue. Descartes, for example, employed the numerical conception of ratios in his *Géométrie* without comment,[15] but

15. Grosholz observes that "Descartes's treatment of ratios and proportions clearly belongs to this [numerical] tradition. In his exposition of multiplication on the second page of the *Geometry*, for example, he infers from the proportion BA:BD :: BC:BE that BE × BA = BD × BC without further comment; there is no need to worry about the association of terms in a non-continuous proportion, since their terms and their ratios

other adherents of the theory felt the need to argue for its acceptance. Wallis upheld the numerical theory and took it as part of his philosophical program to make arithmetic the foundation of all mathematics. Indeed, the numerical theory of ratios is tantamount to an assertion of the priority of arithmetic over geometry; at the very least, the project of interpreting all of mathematics as essentially arithmetical will be helped along by reducing the entire theory of ratios (which traditionally had been deemed quintessentially geometrical) to arithmetic. This is obviously part of Wallis's strategy when he asserts that the comparison of magnitudes in ratios renders all ratios homogeneous and places them in the genus of number, so that the doctrine of ratios belongs more to arithmetic than to geometry.[16] The homogeneity of terms desired by Wallis is guaranteed by his assumption that both rational and irrational magnitudes are truly numbers, so that a ratio is simply a number expressing a numerical quotient of its two terms. This aspect of Wallis's treatment of number led Jacob Klein to describe him as the man whose theories are "the final act in the introduction of the new 'number' concept" (Klein 1968, 211). This new concept of number treats all magnitudes as numbers, whereas the Greek doctrine of number (ἀριθμός) is restricted to positive integers or "collections of units."

Barrow argued for the relational theory as the best alternative to the "corruptions" introduced by the numerical theory. In fact, he devotes the last seven of his twenty-three *Lectiones Mathematicae* to an exhaustive analysis and defense of the Euclidean theory against all objections urged on behalf of contemporary rivals. He acknowledges that his veneration for the classical account of ratios was something of a minority view, but he was convinced that Euclid's presentation of the relational theory was the only philosophically defensible account of ratios. As he explains:

> Nevertheless I must dare to oppose so many and such great men and contradict [ἀντιβλέπειν] such an illustrious authority. Truth needs some sort of protection against such powerful enemies (at

(quotients) and products are all homogeneous in a strong sense. Of course, Descartes chooses line segments, not numbers, to be his homogeneous set of terms" (1991, 53).

16. Wallis says, "But where a comparison of quantities according to ratio is made, it frequently happens that the ratio of the compared quantities leaves the genus of magnitude of the compared quantities and is transferred into the genus of number *[transit in genus numerosum]*, whatever that genus of the compared quantities may be. . . . And this is the principal reason I affirm that the doctrine of ratios belongs rather to the speculations of arithmetic than geometry" (*MU* 25; *OM* 1:136).

least in my judgment). This opinion surely seems to be to be not only false, but wholly harmful, because it engenders and fosters a good many useless controversies and introduces false reasonings [ἀκυρολογίας] and errors into the doctrine of proportions. A few quarrels will be checked, I judge, some difficulties removed, errors avoided, and clouds dispersed, by maintaining that a ratio is not a genus of quantity, nor something subject to quantity, and that it is by no means ever properly attributed to quantity directly and in itself, except by catachresis [κατάχρησιν] or metonymy [μετωνυμίαν]. (*LM* 20, 318)

Barrow's case is simplicity itself: because a ratio is a "pure, perfect relation" it cannot "pass into another category and become a genus of quantity" or be the subject of quantity. He admits that the relational theory of ratios permits such expressions as "the ratio *A*:*B* exceeds the ratio *C*:*D*," but insists that this can be understood without making ratios magnitudes. On Barrow's analysis, one ratio is greater than another when the antecedent of the one is greater than the antecedent of the other, *provided that the ratios have common consequents*. Similarly, the addition and subtraction of ratios proceeds on the assumption that the ratios have been expressed with common consequents.[17] Barrow's attempted defense of the orthodox Euclidean point of view ultimately failed to win the day, as the numerical conception of ratios displaced the older tradition. Nevertheless, it is enlightening to read Hobbes's comments on the nature of ratios and proportions against the background of this dispute.

The Hobbesian theory of ratios is expressed most completely in chapter 13 of *De Corpore*, which bears the title "Of Analogism, or The Sameness of Ratio." This treatment depends upon a brief definition of the term *ratio* in chapter 11, where Hobbes declares:

the similarity, dissimilarity, equality, or inequality of any body to any other body is called its relation; and these bodies are therefore called relata or correlata. Aristotle calls them τὰ πρός τι, the first of which is usually called the antecedent and the second the consequent. But the relation of the antecedent to the consequent

17. Barrow writes, "Whatever is commonly attributed to ratios, only truly and properly agrees with the denominators of ratios, that is, to their antecedents reduced to a common consequent. The quantity that others assign to ratios is nothing other than the quantity and ratio of the denominators, and when they think they add or subtract ratios themselves, they only add or subtract these denominators, and it is the same thing when they multiply or compound, divide or resolve them" (*LM* 20, 325).

according to magnitude, that is to say its equality, excess, or defect, is called the ratio and proportion of the antecedent to the consequent; and so ratio is nothing other than the equality or inequality of the antecedent compared to the consequent according to magnitude. (*DCo* 2.11.3; *OL* 1:118–19)

The most salient feature of this definition is that it takes bodies to be the terms of any relation or comparison, and hence it assumes that all ratios are, in essence, ratios of bodies. This should come as no surprise, given what we have already seen of Hobbes's materialistic treatment of mathematics, but it is worth observing the thoroughness with which Hobbes pursued his program for a philosophy of mathematics based on the concepts of body and motion.

Having defined ratios as relations of magnitudes (whether relations of excess, defect, or equality), Hobbes draws a further distinction between two fundamentally different kinds of ratios, grounding the distinction in two different ways of comparing the excess or defect of one quantity to another. The *arithmetical* ratio of two quantities is formed by considering the simple difference between them, while the *geometrical* ratio of two quantities compares them as part to whole.[18] For example, 5:7 and 11:13 are the same arithmetic ratio because the consequents differ from the antecedents by two; in contrast, 1:3 and 4:12 are the same geometrical ratio because in each case the antecedent is one-third of the consequent. Because geometrical ratios are of much greater interest in the tradition, the term *ratio* is usually taken to be synonymous with *geometrical ratio* and Hobbes generally follows this usage.

Hobbes concludes from this treatment that ratios of excess or defect (whether arithmetical or geometrical) are quantities, while the ratio of equality is not a quantity. The reason for this somewhat unusual view is quite simple: "a ratio of excess, as well as of defect, is a quantity, since it is capable of greater and less; but a ratio of equality, because it

18. Hobbes writes:

> Further, every one of these ratios is twofold. For if it is asked how great any given magnitude is, it can be replied by two ways of comparing it. First, if it is said that it is greater or less than another magnitude by so much, as seven is less than ten by three units. And this ratio is called arithmetical. Second, if it is said that it is greater or less than another magnitude by such a part or parts of it, as seven is less than ten by three tenths parts of the same ten. And although this ratio cannot always be expressed in numbers, it is nevertheless a determinate ratio and of a different kind from the former, and it is called a geometric ratio, or more commonly simply a ratio. (*DCo* 2.13.1; *OL* 1:129)

is capable of neither greater nor less, is not a quantity. And so ratios of inequality can be added together, or one subtracted from another, or multiplied together and by numbers, and divided; but not so ratios of equality" (*DCo* 2.13.3; *OL* 1:130). This doctrine did not originate with Hobbes, but is instead something that he adopted from Mersenne. In his *Cogitata Physico-Mathematica*, Mersenne argued (against the opinion of Clavius) that the proportion of equality is of no quantity, while the proportion of a greater to a lesser magnitude is a genuine quantity, and the proportion of a lesser to a greater is an *antiens* (literally, anti-being) or a quantity less than nothing.[19] Hobbes makes no claim for originality on this point and admits that he first heard of the doctrine from Mersenne during a conversation in Paris while the Minim friar was preparing his *Cogitata Physico-Mathematica* for the press.[20] Wallis found such a doctrine absurd, and in addition to dismissing Hobbes's views as nonsense he was moved to append a brief polemic against Mersenne to his 1657 critique of Marcus Meibom's dialogue on proportions.[21]

The centerpiece of Hobbes's conception of ratios is his alternative definition of sameness of ratio or proportionality.[22] Where Euclid had

19. The doctrine is expressed in the general preface to the *Cogitata*, where Mersenne argues that "[t]he proportion of equality bears a similitude to nothing, a proportion greater than equality is raised above nothing and is similar to being, while the proportion less than equality is pushed down below nothing and can be compared to anti-being [*antienti comparari potest*]" (Mersenne 1644, general preface, sect. 14).

20. As Hobbes says in the *Six Lessons*, "The first time I heard it argued, was in *Mersennus* his Chamber at *Paris*, at such time, as the first volume of his *Cogitata Physico-Mathematica* was almost printed: In which, because he had not said all he would say of Proportion, he was forced to put the rest into a Generall Preface; which as was his custom, he did read to his Friends, before he sent it to the Press" (*SL* 3; *EW* 7:235).

21. Thus, in *Due Correction* Wallis remarks that

> [y]ou tell us, Chap. 13 art. 3 *That the proportion of Inequality is Quantity, but that of Equality is not*. Which *I* said was very absurd; and that the one did no more belong to the Praedicament of Quantity than the other; and that it is to bee, of both equally, either denied or affirmed: And that your argument for it, (That *One equality is not greater or lesse then another; but of proportions of inequality, one may be more or less unequall:*) might as well conclude that *Oblique angles*, be quantities, but not *Right angles*, for these be all equal, and equally Right, but not those. . . . What you alledge out of Mersennus, was but his mistake. (*Due Correction* 68)

The appendix to Wallis's critique of Meibom bears the title "A Criticism of a Passage in Mersenne" (*OM* 1:289–90) and makes essentially the same argument.

22. Substantial linguistic confusions are inherent in any discussion of this topic, since Hobbes uses the Latin *ratio* and *proportio* indifferently for what are usually called

defined proportions in terms of arbitrary equimultiples of paired terms in two ratios, Hobbes seeks a definition that will establish the theory of proportions on the basis of the concepts of body and motion. These considerations lead him to propound the following definition: "One geometric ratio is the same as another geometric ratio when some cause can be assigned that, producing equal effects in equal times, determines both ratios" (*DCo* 2.13.6; *OL* 1:132).

In taking this idiosyncratic approach to the doctrine of ratios Hobbes attempts to steer something of a middle course between the numerical and relational theories. He rejects Euclid's account of ratios as vague, demanding that the Euclidean "sort of relation between magnitudes" be replaced with a treatment of ratios founded on principles of body and motion. Furthermore, in accepting the principle that ratios (or at least ratios of excess and defect) are quantities, Hobbes rejects a key element in Barrow's defense of the relational theory of ratios. But he also opposes the fundamental thesis of the numerical theory, namely that ratios are quotients. Where Wallis had subsumed the doctrine of ratios within the province of arithmetic by identifying ratios with quotients, Hobbes insists that quotients can be formed only by commensurable magnitudes. As he explains the matter in his *Six Lessons:* "In Lines incommensurable there may be the same Proportion, when nevertheless there is no Quotient; for setting their Symboles one above the other doth not make a Quotient; for Quotient there is none, but in *aliquot parts*. It is therefore impossible to define Proportion universally, by comparing Quotients" (*SL* 3; *EW* 7:241).[23] Here Hobbes plainly allies himself with the classical understanding of quotients. He insists that it is impossible to define ratios as quotients, since this will permit only rational numbers as ratios and leave the "universal" treatment of ratios out of mathematics.[24] He further explains that his own

ratios, and uses *analogismo* or *eadem ratio* for what is normally called proportion. These difficulties persist in his English discussions of this topic. Context, however, usually makes Hobbes's declarations unambiguous. I use the terms "ratio" and "proportion" in their usual sense in discussing Hobbes's doctrines and translating his Latin, even where this does not match his English usage.

23. A number K is said to be an aliquot part of a number M, when for some integer n, $M = Kn$. As Hobbes uses the term, he is not claiming that only integers can be quotients, but rather that a quotient can be formed only when the numerator and denominator have "aliquot parts," i.e., when both can be expressed as multiples of a common unit.

24. Thus, Hobbes can retort to Wallis:

> This incommensurability of Magnitudes was it that confounded *Euclide* in the framing of his Definition of Proportion at the fifth Element. For when he came to numbers, he defined the *same Proportion* irreprehensibly thus, *Numbers are*

account of ratios and proportions was developed with the intent of accommodating incommensurable magnitudes within a theory that avoids the obscurity of Euclid while retaining sufficient generality to permit ratios of incommensurables. Thus, Hobbes declares:

> I thought it necessary to seek out some way, whereby the Proportion of two Lines, Commensurable, or Incommensurable, might be continued perpetually the same. And this I found might be done by the Proportion of two Lines described by some uniform motion, as by an Efficient cause both of the said Lines, and also of their Proportions. Which motions continuing, the Proportions must needs be all the way the same. And therefore I defined those Magnitudes to have the same Geometricall Proportion, *when some cause producing in equall times, Equall Effects, did determine both the Proportions.* (*SL* 3; *EW* 7:241–42)

This treatment of ratios was one of the principal targets of Wallis's attack, and I will be concerned in chapter 4 with a more extensive analysis of that part of the controversy that concerns the nature of ratios. For present purposes it is sufficient to observe that one key charge that Wallis levels against Hobbes's theory is that it defines ratios strictly in terms of the difference between two quantities, and thus restricts all ratios to the case of arithmetical ratios. When Hobbes declares in article 5 of chapter 11 of *De Corpore* that "the ratio of an antecedent to a consequent consists in the difference, that is in that part of the greater by which it exceeds the less, or in the remainder of the greater, having taken away the lesser; but not in itself simply, but as it is compared with either of the things related" (*DCo* 2.11.5; *OL* 1:119), Wallis uncharitably reads this as defining ratios by the subtraction of a greater magnitude from a lesser. He then concludes that (according to Hobbes) "wherever there is the same excess or defect, there is the same ratio: and vice versa, wherever there is not the same excess or defect, there is not the same ratio" (*Elenchus* 15). From this Wallis concludes that Hobbes's doctrines can only account for arithmetical ratios. In fact, however, Hobbes does make the traditional distinction

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then Proportional, when the first of the second, and the third of the fourth are equimultiple, or the same part, or the same parts; and yet there is in this Definition no mention at all of a Quotient. For though it be true that if in dividing two Numbers you make the same Quotient, the Dividends and the Divisors are Proportionall, yet that is not the Definition of the same Proportion, but a Theoreme Demonstrable from it. But this Definition *Euclide* could not accommodate to Proportion in Generall, because of incommensurability. (*SL* 3; *EW* 7:241)

between arithmetical and geometrical ratios, and Wallis's complaint on this point derives from a deliberate misrepresentation intended to make his opponent's doctrine look more outlandish than it actually is. The crucial point is the manner in which the excess or defect is compared to the quantities related, and since (as Hobbes observes) there are two ways to effect the comparison, there are two kinds of ratio.

It is ironic that even Barrow, who shared Hobbes's distaste for the numerical theory of ratios, found the Hobbesian conception of ratios thoroughly objectionable and subjected it to an extensive critique in his *Lectiones Mathematicae* (*LM* 23, 370–73). Barrow was particularly bothered by the reliance upon physical concepts in Hobbes's account, and declared "if anyone should read through the writings of all mathematicians (both ancient and recent), I think he would never come across anything an author undertakes to illustrate that is made more obscure, and nothing laboring under more or graver errors" (*LM* 23, 371). Interestingly, Barrow attributes the shortcomings in Hobbes's doctrines to an excessive infatuation with the doctrines of Galileo. According to Barrow, Hobbes "seems more intent upon those things Galileo has written concerning uniform motion, or is fixed upon the contemplation of physical motions, when he refers everything concerning magnitude and quantity to certain preconceived ideas on motion" (*LM* 23, 373). The connection between Hobbes and Galileo will be of greater concern to us when we discuss the third part of *De Corpore*, and particularly the sixteenth chapter. For now, however, we can leave the issue of ratios aside and move on to a consideration of Hobbes's account of figures and angles.

3.3 FIGURE AND ANGLE ACCORDING TO HOBBES

We have seen that Hobbes's materialism leads him to view geometry as a science whose objects are produced by motion and extension, and we have considered some of his proposals to replace the traditional definitions with ones that treat geometry as a science of body. This campaign extends to cover the definition of such important geometric objects as lines, figures, and angles; in this section I investigate Hobbes's attempts to define such objects.

The place to start is with the definition of a line. We have seen that Euclid's definition declares a line to be "breadthless length" (*Elements* 1, def. 2). This definition was taken to apply both to straight and curved lines, so a special definition of straight (or right) line is needed. Euclid defines a right line as "a line which lies evenly with the points on

itself" (*Elements* 1, def. 4)—a definition notorious for its obscurity.[25] Hobbes dismisses it as "inexcusable" and wonders "How bitterly, and with what insipide jests, would [Wallis] have reviled *Euclide* for this, if living now he had written a Leviathan" (*SL* 1; *EW* 7:203). He therefore offers his own definition in terms of motion, but not simply the motion of a point. He defines a right line as a special case of a line, namely one whose termini cannot be understood to be drawn apart without altering its magnitude. The magnitude of a line is "judged by the greatest distance that can be between its termini" (*DCo* 2.14.1; *OL* 1:154), so the image here is that a straight line cannot have its termini drawn farther apart, but the endpoints of a curved line can be separated while the line retains the same length. The result is that we must consider two kinds of motions in defining a right line: first the motion of a point (by which a line *simpliciter* is traced), and then a motion drawing the termini of the line apart from one another. If the second kind of motion cannot be conceived without altering the magnitude of the line, it is a right line.

We have already seen that Hobbes defines surfaces and solids in terms of the motions of bodies. The natural extension of this doctrine is the definition of plane figures as quantities determined within a plane, or a portion of a surface determined by extreme points or boundaries. As Hobbes defines it: "a figure is the quantity determined by the location or position of all its extreme points" (*DCo* 2.14.22; *OL* 1:174). The determination of these extreme points can be carried out in any number of ways, but the motion of lines and points within a surface is obviously the principal means of such determination of figures. It is important to observe that Hobbes does not define a surface or plane figure as the product of two lines; in his view, the "drawing of lines into lines" is a quintessentially geometric operation that must not be conflated with the arithmetical operation of multiplication. Although the area of a rectangle can be computed by multiplying the lengths of its sides, this is not the process by which the figure is actually created and therefore cannot be part of its definition.

A figure of special interest is the circle, which Euclid defines as "a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another" (*Elements* 1, def. 15). Hobbes agrees that this defini-

25. Heath's commentary (Euclid [1925] 1956, 1:165–69) gives a good overview of the difficulties with the definition and suggested alternatives, which can be supplemented by Federspiel 1991.

tion tells us something true about the circle, but argues that it is not adequate as a definition because it fails to give a causal account of its generation. As he puts it in the first of his *Six Lessons:*

> But if a man had never seen the generation of a Circle by the motion of a Compass or other aequivalent means, it would have been hard to perswade him, that there was any such Figure possible. It had been therefore not amiss first to have let him see that such a Figure might be described. Therefore so much of Geometry is no part of Philosophy, which seeketh the proper passions of all things in the generation of the things themselves. (*SL* 1; *EW* 7:205)

On Hobbes's account, the proper definition of a circle is in terms of the rotation of a line about one of its termini.[26] This definition not only gives the cause of the circle (which can be shown by the use of a compass) but thereby enables us to investigate its properties. Hobbes's methodology holds that once the cause of something has been identified, it is possible to establish demonstrative truths about that thing, and there is nothing that need remain hidden or unexplained. In fact, it is Hobbes's faith in such definitions by causes that led him to think that such notorious problems as the quadrature of the circle could be easily solved once one had grasped the nature of the circle. Hobbes specifically mentions knowledge of the properties of the circle as an example of how "the knowledge of any effect may be gotten from its known generation" (*DCo* 1.1.5; *OL* 1:6). According to Hobbes, it would be impossible to tell by sense whether a figure that closely resembles a circle is, in actuality, a circle or (say) an ellipse that differs very minutely from a true circle. But if we are told that the figure was

26. In *De Corpore*, the definition reads, "If a right line in a plane is so moved that, while one endpoint remains in place, the whole line is carried about until it returns to the place from which it began to move, a plane surface terminated in every direction will be described by the curved line that is described by that endpoint of the line that was carried around. This surface is called a circle, and the unmoved point is called the center of the circle, and the curved line that terminates it is called the perimeter, and any part of it is called the circumference or arc" (*DCo* 2.14.4; *OL* 1:157). It might be objected that this definition presupposes a good deal of knowledge of the circle, since it implicitly assumes that the line, in being "rotated" or "carried about" will return to the same position in which it started. Indeed, to define the circle in terms of its generation *by circular motion* looks plainly circular, if such a bad pun can be excused. Curiously, although Wallis raised any objection he could think of to Hobbes's definitions, he could only fault the definition of the circle for requiring the point at the center of the circle remain unmoved: "But if it is unmoved, it is not a point, since you defined a point (in chapter 8, article 12) as 'a body so moved,' etc." (*Elenchus* 26).

generated by the circumduction of a line about one of its endpoints, there will be no difficulty in identifying it as a genuine circle. This identification of the cause will therefore enable other effects or properties of the circle's generation (such as its area in comparison with a square) to be known.

Angles are the final class of geometric objects to consider. Hobbes was of the opinion that the understanding of angles requires a prior understanding of the circle, and he was convinced that his principles allow for an account of angles that is vastly superior to that in Euclid. Euclid defines an angle as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line" (*Elements* 1, def. 8). Hobbes dismisses this definition as both vague and poorly understood.[27] To overcome the difficulties he finds with the Euclidean definition, he proposes an alternative. The first step here is to recognize that the classical account of angles allows for angles to be formed by the concourse of right lines as well as curves. Thus, all of the examples in figure 3.1 could be classified as angles in the Euclidean sense. These include the familiar case of rectilinear angles, curvilinear angles formed by two curves, and the "mixtilinear" case formed by the concurrence of a right line and a curved line. Hobbes was aware of problems posed by the case of some angles involving curves, and it is clear that he fashioned his account of angles to avoid these difficulties. I will have more to say about these matters in chapter 4 when I investigate the problem of the angle of contact, but for now it is sufficient to note that in *De Corpore* Hobbes begins his discussion of angles by drawing a distinction between lines (both curved and right) that touch one another and those that cut one another. As he phrases the distinction:

> Two lines are said to *touch* one another if, when they are drawn to the same point, and produced any amount (produced, I say, in the same manner by which they are generated), still will not cut one another. And so two right lines, if they touch one another at any point, will be contiguous throughout their whole length. (*DCo* 2.14.6; *OL* 1:159)

One line cuts another if it falls on both sides of it, and this distinction thus introduces two different ways in which lines that form angles can meet in a point.

27. As Hobbes writes in *De Principiis et Ratiocinatione Geometrarum*: "Few have impressed in their minds the idea or image of the angle, of which so much is said by

Figure 3.1

Hobbes argues that the most general definition of the term *angle* depends upon the fact that angles are formed where lines share a common point. As he phrases it, the "most general" definition of the term is "*the quantity of that divergence, when two or more lines or surfaces come together in a single point and diverge everywhere else*" (*DCo* 2.14.7; *OL* 1:159). We can leave aside the case of angles formed by surfaces and confine our attention to the case of angles in a plane. The general definition of *angle* allows for two species of angles, which are distinguished by "the two ways by which lines may diverge from one another":

> For two right lines applied to one another are contiguous throughout their whole length, but one can be pulled apart from the other, leaving their concurrence in one point, either by circular motion (the center of which is that point of concurrence) so that each line retains its rectitude, and the quantity of the separation or divergence is an *angle* simply so called; or one line may be pulled apart by continual flexion or curving in every imaginable point; and the quantity of this separation is called the *angle of contingence*. (*DCo* 2.14.7; *OL* 1:159–60)

Circular motion is essential to the generation of both kinds of angles, but simple angles arise when the lines forming the angle remain rigid through the separation, while "angles of contingence" originate from the uniform (i.e., circular) flexion of one line as it is pulled away from the other. Observe also that in the first case the lines that form the angle cut one another, while in the second they do not.

The quantity of a simple angle can be measured from the amount of rotation of a circle's radius, as in the following example. Let *AB* in

geometers. For the most part, whatever in a surface is open wide at one end, and ends in a narrow end, is commonly called an angle" (*PRG* 8; *OL* 4:399).

figure 3.2 be a radius of circle *BCD*, and let *E* be any point on *AB*. Consider the circles *BCD* and *EFG*. We note that if an arbitrary point *H* is taken on *BCD*, $\widehat{BH}$ and $\widehat{EI}$ will be formed. The angle $\angle BAH$ is clearly of the same magnitude as $\angle EAI$, but the measure of the angle cannot be defined by the areas of the sectors *AEI* and *ABH* or by the size of $\widehat{EI}$ and $\widehat{BH}$. But the circles are generated by the motion of the radius *AB*, and in generating these circles, $\angle BAH$ and $\angle EAI$ are formed. Hobbes's idea is that the angle can be defined in terms of the construction of the circle by declaring the angle to be the amount of "circumlation" or rotation required by radius *AB* to generate the sectors *BAH* and *EAI*. Using this definition, we can take as the measure of the angle $\angle BAH$ the ratio of $\widehat{BH}$ to the circumference of the circle *BCD*, since $\widehat{BH}:BCD :: \widehat{EI}:EFG$.

This account underscores the fact that "neither the length nor the equality or inequality of the lines that comprehend it contributes anything to the quantity of an angle," and leads to the definition that "*the quantity of an angle is the quantity determined by the ratio of the arc or circumference of the circle to the whole perimeter*" (*DCo* 2.14.9; *OL* 1:161). In the example used thus far, we have only considered rectilinear angles, but Hobbes's account of simple angles also includes angles formed by curved lines that cut one another. To measure such angles, it is necessary that the measuring circle be constructed from the tangents to the curved lines at the point where they cut. Thus, if the curve *AB* cuts the curve *CD* at *E* (as in figure 3.3), Hobbes defines the angle at *E* from the tangents $E\alpha$ and $E\beta$. As Hobbes puts it: "the angle that two curved lines make is the same as that which is made by

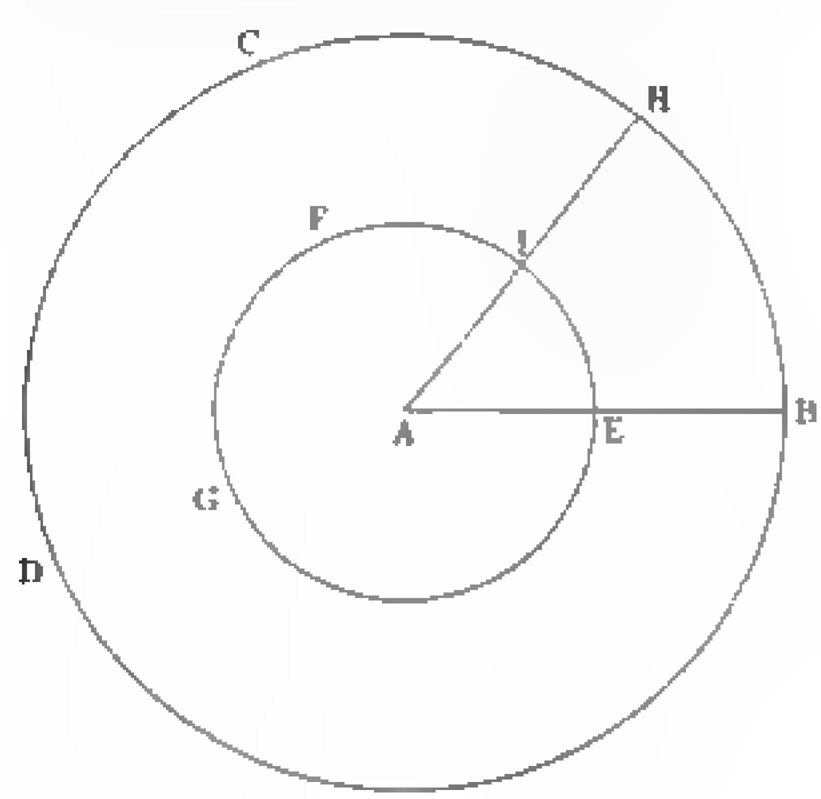

Figure 3.2

two right lines that touch them" (*DCo* 2.14.9; *OL* 1:162). Angles of contingence fail to satisfy this definition, but Hobbes does not conclude that they are not measurable. Instead, he argues that they are heterogeneous to ordinary angles and properly measured in terms of degree of curvature. His definition is: "The angle of contact is the quantity of curvature that is in the arc of the circle made by continual and uniform bending of a right line" (*PRG* 8; *OL* 4:406). To understand this definition, recall that the angle of contact is formed by the intersection of the circle and a right line. If *BCD* is a circle and *BK* a line tangent to the circle (figure 3.4), the angle of contact is the angle *CBK*. Hobbes insists that such angles must be quantities, since the circle can have a greater or lesser divergence from the straight line, and anything that can be ordered by a relation of greater and less must be a quantity. Angles of contact can therefore be compared with one another by observing the amount of curvature in $\widehat{BC}$, since the greater the curvature, the greater the angle. As a measure of this degree of curvature, we can take the inverse of the radius of the circle, since the degree of curvature varies inversely as the radius of the circle *BCD*.[28]

This conception of angles will occupy our attention in chapter 4 when we consider Hobbes's and Wallis's polemics over the problem of the angle of contact between a curve and tangent. For now, however, we can leave this part of the Hobbesian enterprise and move to a study of those chapters of *De Corpore* that are directed at the solution of problems of quadrature and the rectification of curvilinear arcs.

3.4 MAGNITUDE, MOTION, AND THE GENERAL METHOD OF QUADRATURE

Hobbes intended his program for mathematics to do more than simply substitute one set of definitions for another. Definitions are certainly

28. As Hobbes puts it:

> Since the angle in general is defined as the opening or divergence of two lines that come together in a point, and since one opening is greater than another also in the generation of the angle of contingence, it cannot be denied that this angle is a quantity, for wherever there is greater and less, there is also quantity. But this quantity consists in greater and lesser flexion. For as the circumference of the circle is greater, its circumference approaches the nature of a right line; inasmuch as the whole curvature (which is made when the periphery of a circle is traced by a right line) is greater when applied to a lesser right line. And therefore, when many circles are tangent to one right line, the angle of contingence made with a lesser circle is greater than that made with a greater circle. (*DCo* 2.14.16; *OL* 1:170)

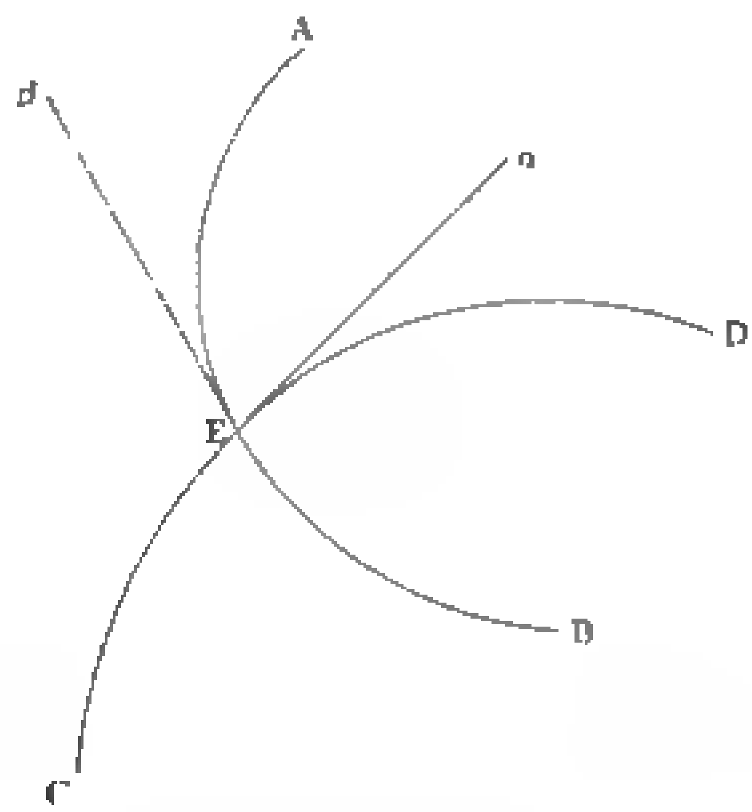

Figure 3.3

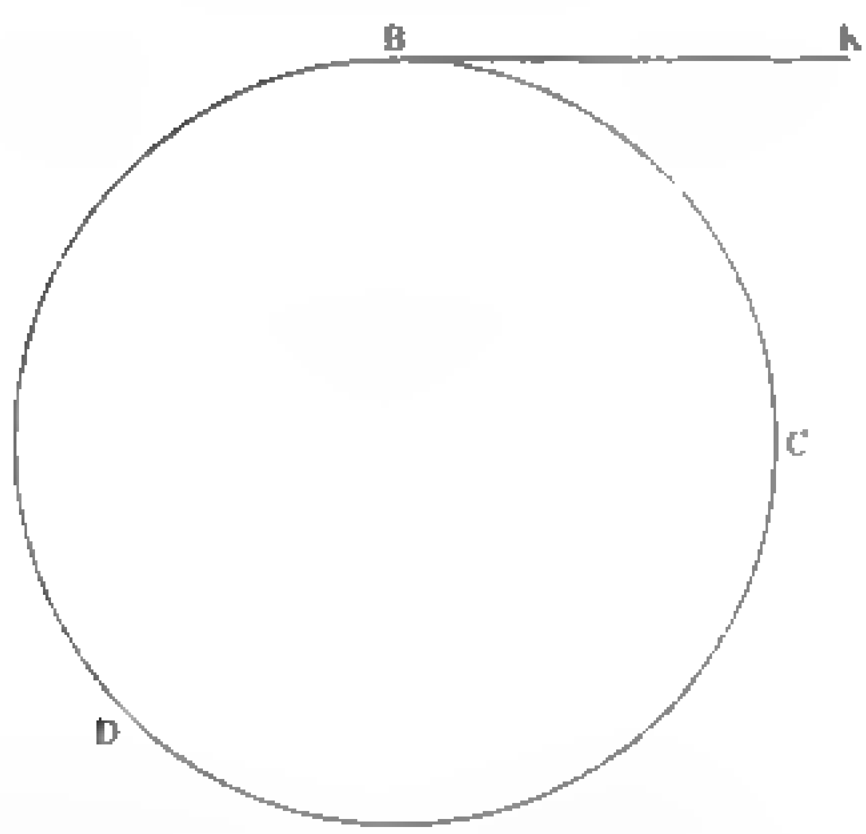

Figure 3.4

important, but the purpose of Hobbes's rewriting of Euclid was to enable the solution of outstanding problems and the resolution of controversies.[29] The fruits of Hobbes's definitional efforts were supposed to be put on display most prominently in the third part of *De Corpore* (comprising chapters 15 through 24), which bears the title "On Ratios

29. Du Verdus was enthusiastic in his praise for Hobbes's definitions. It must have pleased the philosopher greatly to read:

> Before your definitions appeared, one of two things happened: either people did not understand the matter at all, or they just talked gibberish about it. For it was quite possible, for example, to know what a straight line was; but as for having a clear and distinct idea corresponding to the words "which lies evenly with the

of Motions and Magnitudes" and contains Hobbes's presentation of a general method of investigating ratios between magnitudes by considering the motions through which such magnitudes are generated. In the opening article of the fifteenth chapter, Hobbes announces that "I thought I should advise the reader coming to this place that he should take into his hands the writings of Euclid, Archimedes, Apollonius, and others, both ancient and modern. For why should I do what has already been done? The little I will therefore say of geometric matters in some of the following chapters will be new, and such as is chiefly of service to natural philosophy" (*DCo* 3.15.1; *OL* 1:175–76). Not surprisingly, part 3 also contains the Hobbesian attempt to square the circle, but these chapters also have a close relationship to Galileo's analysis of uniformly accelerated motion and to Cavalieri's presentation of the method of indivisibles. Indeed, this part of *De Corpore* clearly shows Hobbes's debt to the scientific and mathematical work of seventeenth-century Italian sources. In this section I outline the Hobbesian doctrines derived from Galileo and Cavalieri; more particularly, I will concentrate on chapters 15–18 of *De Corpore* as well as their relationship to Galileo's *Two New Sciences* and Cavalieri's *Exercitationes Geometricae Sex*. I will also investigate the connection between Hobbes's methods and those of Roberval, particularly in connection with the problem of determining the arc length of the Archimedean spiral.

The key to Hobbes's approach to the general problem of quadrature in part 3 of *De Corpore* is his concept of *conatus*.[30] As Hobbes defines it, *conatus* is essentially a point motion, or motion through an indefinitely small space: "*conatus* is motion through a space and a time less than any given, that is, less than any determined whether by exposition or assigned by number, that is, through a point" (*DCo* 3.15.2; *OL* 1:177). Naturally, Hobbes employs his own conception of points here, and in explicating the definition he remarks that "it should be recalled

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> points on itself", I for one had no corresponding idea at all. But when you write, it is Nature herself who is speaking. Above all, Sir, you have the Honour, which none can take away from you, of having been the first true founder of geometry. You are its creator. (du Verdus to Hobbes 13/23 December 1655; *CTH* 1:227)

30. The term *conatus* derives from the Latin verb *conor,* meaning to strive or attempt. The English translation of *De Corpore* uses the term "endeavour," which was apparently Hobbes's preferred English equivalent. Nevertheless, I will retain the Latin usage, as it has gained wide currency. For studies of this doctrine see Barnouw 1992; Brandt 1927, chap. 9; and Lasswitz 1890, 2:214–24. Bernstein 1980 studies Hobbes's conception of *conatus* in relation to Leibniz's early doctrines of motion.

that by a point is not understood that which has no quantity, or which can by no means be divided (for nothing of this sort is in the nature of things), but that whose quantity is not considered, i.e., neither its quantity nor any of its parts are computed in demonstration, so that a point is not taken for indivisible, but for undivided. And as also an instant is to be taken as an undivided time, not an indivisible time" (*DCo* 3.15.2; *OL* 1:177–78).

This definition allows for a further concept of *impetus*, or the instantaneous velocity of a moving point; the velocity of the point at an instant can be understood as the ratio of the distance moved to the time elapsed in a *conatus*. In Hobbes's terms "*impetus is this velocity* [of a moving thing] *but considered in any point of time in which the transit is made*. And so impetus is nothing other than the quantity or velocity of this *conatus*" (*DCo* 3.15.2; *OL* 1:178).[31]

The concepts of impetus and *conatus* can be applied to the case of geometric magnitudes as well as to moving bodies. Because magnitudes are generated by the motion of points, lines, or surfaces, it is possible to inquire into the velocities with which they are generated, and this inquiry can be extended to the ratios between magnitudes and their generating motions. For example, we can think of a curve as being traced by the motion of a point, and at any given stage in the generation of the curve, the point will have a (directed) instantaneous velocity. This, in turn, can be regarded as the ratio between the indefinitely small distance covered in an indefinitely small time; this ratio will be a finite magnitude that can be expressed as the inclination of the tangent to the curve at a point. Take the curve $\alpha\beta$ as in figure 3.5. The *conatus* of its generating point at any instant will be the "point motion" with which an indefinitely small part of the curve is generated; the impetus at any stage in the curve's production will be expressed as the ratio of the distance covered to the time elapsed in the *conatus*. If we represent the curve in the familiar coordinate axis system, the instantaneous impetus will be the ratio between the instantaneous increment along to the y-axis to the increment along the x-axis. The tangent to the curve

31. The English version of this passage is significantly different: "I define IMPETUS, *or Quickness of Motion, to be the Swiftness or Velocity of the Body moved, but considered in the several points of that time in which it is moved; In which sense* Impetus *is nothing else but the quantity or velocity of Endeavour. But considered with the whole time, it is the whole velocity of the Body moved, taken together throughout all the time, and equal to the Product of a Line representing the time, multiplyed into a Line representing the arithmetically mean* Impetus *or Quickness*" (*EW* 1:207). These modifications allow Hobbes to distinguish the instantaneous impetus of a body at a point from the entire impetus acquired over an interval of time.

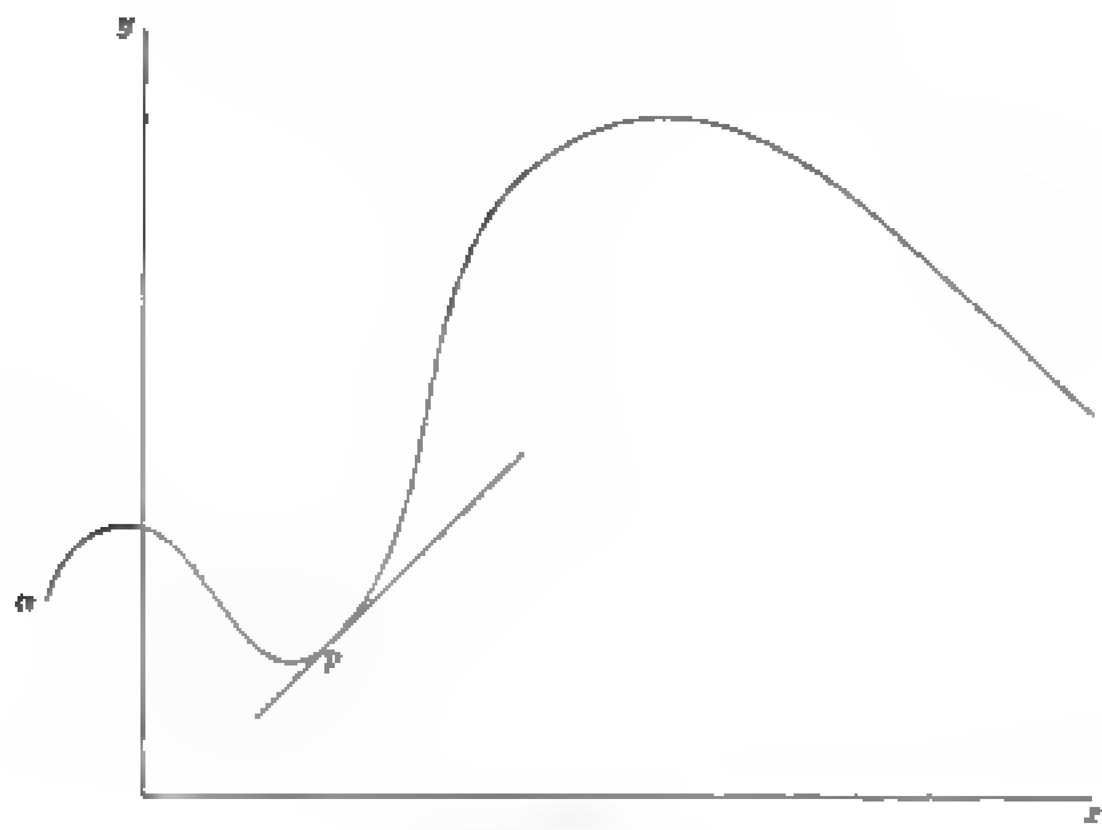

Figure 3.5

at the point *p* can then be analyzed as the right line that continues or extends the *conatus* at *p* (equivalently, the tangent is the dilation or expansion of the point motion into a right line).[32]

This approach can also be extended to the general problem of quadrature, with the area of a plane figure being analyzed as the product of a moving line and time. In the very simplest case, the whole impetus imparted to a body throughout a uniform motion will be representable as a rectangle, one side of which is the line representing the instantaneous impetus while the other represents the time during which the body is moved. More complex cases can then be developed by considering nonuniform motions produced by variable impetus. Hobbes applies the concept of impetus to problems of quadrature in chapters 16 and 17 of *De Corpore*. In chapter 16 he considers uniform and accelerated motions of bodies, while in chapter 17 he deals with "deficient figures" produced by the motion of lines. Both of these chapters are of great interest, not for the quality of the mathematical work (which is often sloppy and occasionally quite inept), but because they show the sources of Hobbes's conception of mathematics and its method.

3.4.1 Galileo, Hobbes, and Chapter 16 of De Corpore

As I have indicated, Galileo's work on naturally accelerated motion is prominent as part of the background for Hobbes's analysis of motion

32. There is a significant similarity between this approach and Leibniz's formulation of the infinitesimal calculus. I explore the possibility of a connection between Hobbes's mathematics and Leibniz's thoughts on the calculus, via the concept of *conatus*, in Jesseph 1999a.

and magnitude. In fact, it is fair to consider chapter 16 as Hobbes's attempt to incorporate Galileo's work into his own program for a thoroughly materialistic mathematics. Galileo's treatise *Two New Sciences* is devoted to the exposition of a mathematical analysis of the science of mechanics and the science of local motion. In particular, the Galilean treatment of local motion investigates naturally accelerated motion by applying the concept of impetus and taking acceleration to be the accumulation of successive impetuses over time, so that in uniformly accelerated motion the increment in speed is proportional to the increment in time. Galileo defines uniformly accelerated motion in his *Two New Sciences* to be that "which, abandoning rest, adds on to itself equal momenta of swiftness in equal times" (Galileo 1974, 162).[33] The analysis of uniform acceleration leads to two key theorems, the first of which is the "mean speed theorem":

> The time in which a certain space is traversed by a moveable in uniformly accelerated movement from rest is equal to the time in which the same space would be traversed by the same moveable carried in uniform motion whose degree of speed is one-half the maximum and final degree of speed of the previous, uniformly accelerated, motion. (Galileo 1974, 165)

Galileo's proof of this theorem begins by representing the time by the line *AB* and the space traversed by the line *CD* as in figure 3.6. The line *EB* drawn perpendicular to *AB* represents the final velocity of the moving body, and the parallels between the lines *AB* and *AE* represent instantaneous velocities of the body, increasing uniformly along the time line *AB*. The triangle *ABE* thus represents the accumulated impetus of the accelerated motion. Bisecting *BE* at *F* and drawing *FG* parallel to *AB* yields a rectangle *ABFG* whose area is equal to that of the triangle *ABE*. This rectangle can be thought of as representing the motion of a body, which begins with a velocity one-half that of *AE*, and continues unaccelerated throughout the time *AB*. The distance covered will be equal in both cases, since that traveled by the unaccelerated body is the product of velocity and time, while that of the accelerated body is treated as the aggregate of instantaneous impetus over the same time. Galileo does not want to assert that the areas of the rectangle and triangle actually represent the distances, since the distance is

33. The expression *momenta of swiftness* is equivalent to Hobbes's term *impetus*. As Stillman Drake notes, *impetus* is "freely interchangeable with *momento*" (Galileo 1974, xxxiv).

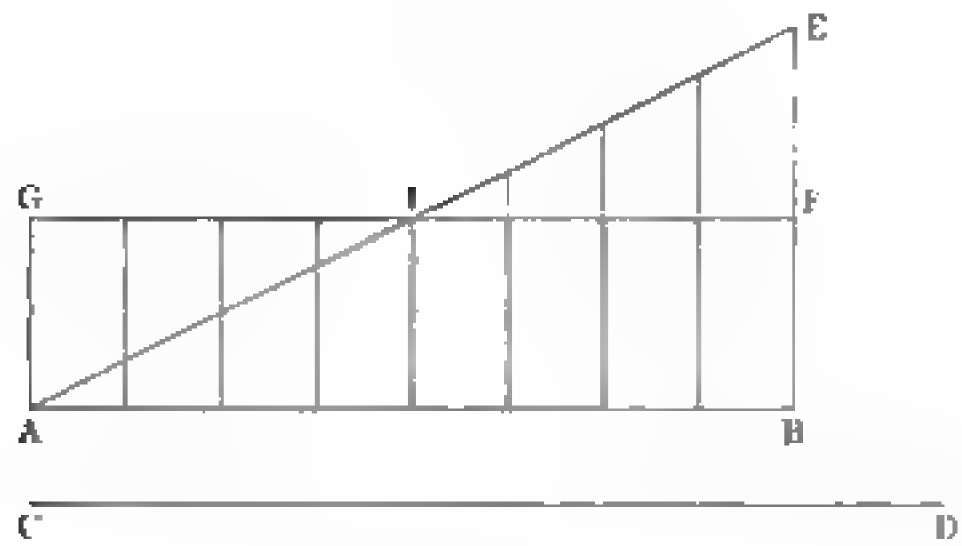

Figure 3.6

supposed to be given by the line *CD* (which cannot be compared to an area). Nevertheless, he can assert that the ratio between the distances is the same as that of the areas, and this suffices for the theorem.

The second key theorem in Galileo's account of accelerated motion relates the distance traversed to the elapsed time, and establishes that the ratio between any two distances covered by accelerated motion is the same as the ratio of the squares of the times. In Galileo's words: "If a moveable descends from rest in uniformly accelerated motion, the spaces run through in any times whatever are to each other as the duplicate ratio of their times; that is, as the squares of those times" (Galileo 1974, 166). The details of the proof need not detain us here,[34] for the important point is the connection between Hobbes's and Galileo's approaches to the mathematics of motion. As we will see, Hobbes is eager to exploit this result as part of his program in the philosophy of mathematics.

A final Galilean result of importance for understanding chapter 16 of *De Corpore* is the theorem that a projectile traces a parabolic path. We can follow Galileo in understanding a projectile as "a moveable [whose] motion is compounded from two movements; that is, when it is moved equably and is also naturally accelerated" (Galileo 1974, 217). In particular, projectile motion arises when a body is moved laterally with a uniform velocity, but naturally accelerated downwards. The principal result of interest to us is the first theorem of the fourth-

34. They can be found in Galileo 1974, 166–67. Essentially, the proof invokes the previous result to show that the ratio of the distances at any two points can be compared by considering equivalent distances covered with uniform velocity. Galileo then invokes the principle that the distances covered by equable motion are in the ratio compounded from the speeds and times, from which he concludes that the distances covered by uniformly accelerated motion must be as the squares of the elapsed times. For a detailed study of Galileo's mathematization of naturally accelerated motion see Clavelin 1974, 298–323.

day dialogue in the *Two New Sciences.* It asserts that "[t]he line described by a heavy moveable, when it descends with a motion compounded from equable horizontal and natural falling [motion], is a semiparabola" (Galileo 1974, 221). The parabola is a line with the property that the distance along the vertical y-axis varies as the square of the distance along the horizontal x-axis (to use the terminology of coordinate axes). Galileo's theorem asserts that if we assume the movable to have uniform motion along the x-axis (line *AB* in figure 3.7) and to be naturally accelerated downward along the y-axis (line *AC*), then the combination of these two motions will trace the path *AD,* in which uniform horizontal motion combines with accelerated downward motion increasing as the square of the time. The result is one-half of the parabola $y = x^2$, and the path *AD* is thus a semiparabola.

Scholars have long recognized that Hobbes was influenced by Galileo, and the link between the two thinkers is well established by Hobbes's remark in the "dedicatory epistle" to *De Corpore* that Galileo "was the first who opened to us the first gate of universal physics, which is the nature of motion" (*DCo* epistle; *OL* 1:sig. h4v). Legend even has it that a conversation with Galileo in 1635 or 1636 inspired Hobbes to pursue the goal of presenting moral and political philosophy in a rigorously geometrical method.[35] It is also worth observing

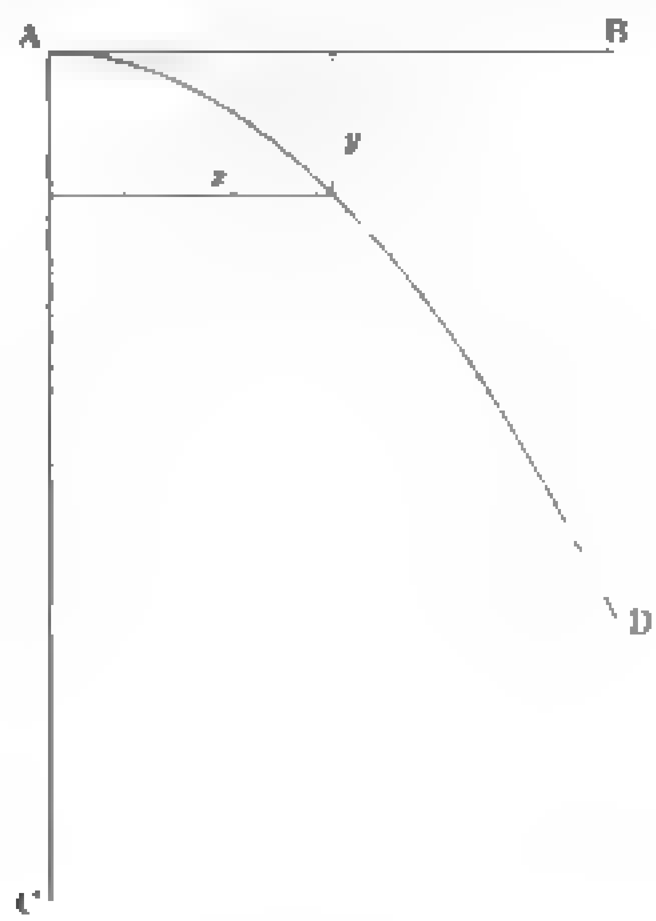

Figure 3.7

35. Tönnies 1975, 85–91, reports the evidence of a personal connection between Hobbes and Galileo. The source for Galileo's alleged direct influence on Hobbes's program in moral and political philosophy is A. G. Kästner's *Geschichte der Mathematik,*

that Mersenne was active in bringing the work of Galileo to the attention of the learned circles in Paris, and Hobbes's exposure to Galileo's thought may well have been mediated by his contact with Mersenne. Whatever its full extent may have been, Hobbes's indebtedness to Galileo is evident in chapter 16 of *De Corpore*, which bears the title "On Accelerated and Uniform Motion, and on Motion by Concourse." The approach pursued here takes uniform motion to be the result of a constant impetus, so that a body moved uniformly traverses equal distances in equal times. The velocity of a body, as Hobbes understands it, "has its quantity determined by the sum of all the several quicknesses or impetus[es], which it hath in the several points of the time of the body's motion" (*DCo* 3.16.1; *EW* 1:218).[36] Representing the motion of a body by a figure, one side of which designates the time and the other the impetus generated in the corresponding instant, we have a rectangle (such as *ABCD* in figure 3.8) in the case of uniform motion. The uniform motion starts with a given impetus *AB*, which remains unaltered as the body moves through time *BC*. On the other hand, motion uniformly accelerated from rest is represented by the triangle *BCD*, in which the impetus increases continually with the time.[37] Finally, the calculation of distances covered by such motions proceeds in essentially the same manner as that used by Galileo—a body uniformly accelerated from rest covers distances that are in the duplicate ratio of the elapsed times, so that the distance dispatched is as the square of the time. (*DCo* 3.16.3; *OL* 1:186–88)

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which reports that "John Albert de Soria, former teacher at the university in Pisa, assures us it is known through oral tradition that when they walked together at the grand-ducal summer palace *Poggio Imperiale*, Galileo gave Hobbes the first idea of bringing moral philosophy *[Sittenlehre]* to mathematical certainty by treating it according to the geometrical method" (Kästner 1796–1800, 4:195).

36. The Latin version of this passage is less clear than the English. At the beginning of chapter 16 Hobbes has the definition: "VELOCITAS cujuscunque corporis per aliquod tempus moti tanta est, quantum est quod fit ex impetu (quem habet in puncto temporis) ducto in tempus ipsius motus" [The velocity of any body moved through some time is equal to the product of the impetus that it has at a point of time and the time in which it is moved] (*DCo* 3.16.1; *OL* 1:184). Wallis attacks this definition for the obvious problem, namely that it does not specify at which point of time the product of impetus and time should be computed (*Elenchus* 39).

37. As Hobbes puts it, "If the impetus is everywhere the same and any right line is taken for the measure of time, the impetuses applied ordinately to this right line will designate a parallelogram, which will represent the velocity of the whole motion. If the impetus increases uniformly from rest, that is, always in the same ratio as the elapsed times, the whole velocity of the motion will be represented by a triangle, one side of which is the whole time, and the other the greatest impetus acquired in that time" (*DCo* 3.16.1; *OL* 1:185).

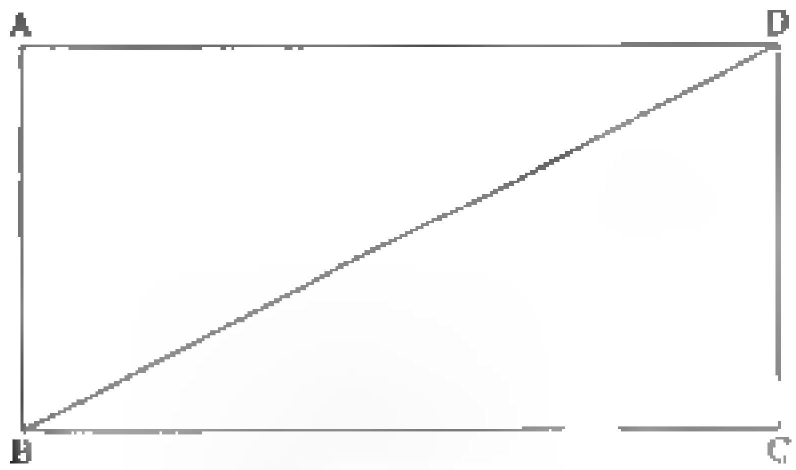

Figure 3.8

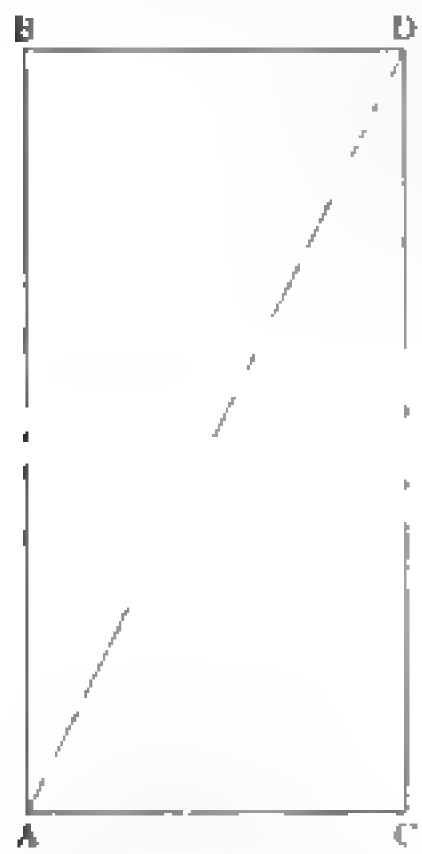

Figure 3.9

The investigation of "motion by concourse" in the remainder of chapter 16 of *De Corpore* is another part of Hobbes's Galilean inheritance. In exact analogy with Galileo's treatment of projectile motion, Hobbes treats motion of a body acted upon by two forces as a complex "concourse" of the two simpler motions. In the simplest case, where the combined motions are both uniform, the resulting motion will be a straight line equal to the diagonal of a parallelogram whose sides are proportional to the two component velocities. Thus, in figure 3.9, the compound motion arising from the concourse of motions *AC* and *AB* will be the diagonal motion *AD*. This result can then be extended to consider cases of nonuniform velocity. In particular, Hobbes asserts that

> [i]f a moving body *[mobile]* is carried by two movements together, meeting at any given angle, of which the first moves uniformly and the other by a motion uniformly accelerated from rest

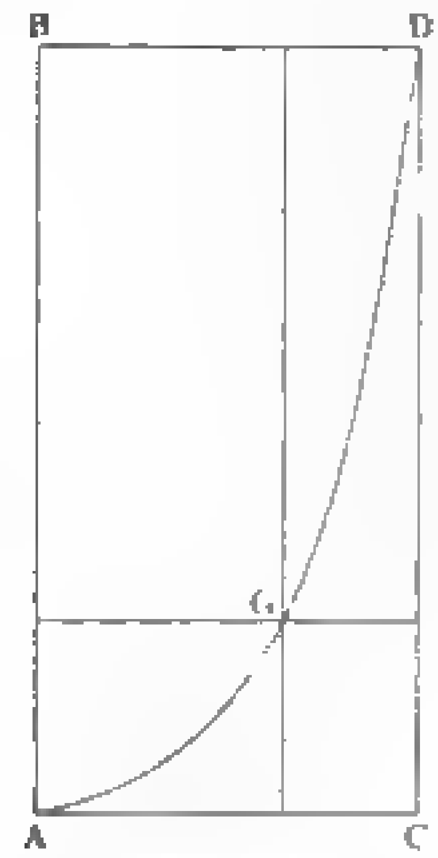

Figure 3.10

> (that is, so that the impetuses are in the ratio of the times; that is, so that the ratio of the lengths is the duplicate ratio of the times) until the greatest impetus acquired by acceleration is equal to the impetus of uniform motion, the line in which the moving body is carried will be a semiparabola, whose base is the impetus last acquired. (*DCo* 3.16.9; *OL* 1:196)

This result is, of course, identical to Galileo's theorem that projectile motion describes a semiparabola, although it is phrased somewhat differently. In this case, the semiparabola *AGD* in figure 3.10 arises from the concourse of the uniform motion *AC* and the accelerated motion *AB*. As Hobbes presents the case, he imagines the straight line *AB* to be moved uniformly to *CD*, while in the same time the line *AC* is moved with uniform acceleration to *BD*. The locus of intersection of the two moving lines will then be the semiparabola *AGD*. This differs slightly from the presentation of projectile motion in Galileo, but is clearly closely related to it.

Although there is an obvious similarity between Hobbes's and Galileo's analysis of motion, there are also important differences. Galileo was concerned with the mathematical analysis of motion, and the results presented in his *Two New Sciences* are confined to such phenomena as arise from naturally accelerated and projectile motion. Hobbes, however, pursues a more general and abstract inquiry in *De Corpore*, for he also considers motions that correspond to nothing in nature. He extends, for example, the case of projectile motion to the case of motion accelerated as the triplicate ratio of the times, and then considers

a general rule for describing and investigating the quadrature of "parabolasters" of higher degree that correspond to accelerations of any degree whatever.[38] This extension of Hobbes's investigations beyond the case of natural motions led Frithiof Brandt to see a certain incongruity between these chapters of *De Corpore* and Hobbes's general program for a thoroughly mechanistic science of nature, and he regarded it as strange that Hobbes should have bothered to consider motions that do not arise in nature.[39] However, there is really no deep difficulty here. Hobbes was clearly concerned to present an account of mathematics that could extend to the most abstract and general classes of problems, and his program for a thoroughly materialistic conception of mathematics would naturally have to extend beyond those parts of mathematics that have an immediate physical application. Notwithstanding his declaration that his contributions to geometry were to be "such as are chiefly of service to natural philosophy" (*DCo* 3.15.1; *OL* 1:176), Hobbes was also concerned with problems in pure mathematics. If nothing else, he was hardly alone in seeking a general approach to problems of quadrature.

Wallis was unrelenting in his critique of Hobbes's sixteenth chapter, and he devoted a remarkable 28 pages of his 136-page *Elenchus* to a detailed criticism of it. Of the twenty articles that initially made up this chapter, he denounced all but three as unsound, for reasons ranging from sloppy formulation of principles to technical errors in Hobbes's demonstrations. Wallis was happy to supply suggestions for

38. As Hobbes puts it, "By the same method it can be shown what line it is that is made by a moving body driven *[actum]* by the concourse of any two motions, of which one is moved uniformly, while the other is accelerated, but according to ratios of spaces and times that are explicable by numbers, such as duplicate, triplicate, and other such ratios, or such ratios as can be designated by any fraction whatever" (*DCo* 3.16.11; *OL* 1:198). Hobbes uses the term *parabolaster* for curves of higher degree than the simple parabola.

39. As Brandt puts it:

> That this is far removed from Galileo, seems to us fairly obvious. The latter did not occupy himself at all with the possibilities posited by Hobbes, of course because they have no relation to experience. Result: *Hobbes both directly and indirectly shows a conspicuous lack of interest in mathematics as applied to the motions occurring in experience. He shows this directly by making no mention whatever of the fact that the formula $s = at^2$ is the law for the free fall and that the parabola of projection is compounded of the uniform motion and the accelerated motion on the assumption $s = at^2$. And he shows it directly by placing the aforementioned possible formulae of motion on an equal footing. This is mathematically warrantable but seems strange in a philosopher who is driving at a mathematico-mechanical conception of Nature.* (Brandt 1928, 320)

the mending of such errors, and he recommended his own *Arithmetica Infinitorum* as a superior treatment of these matters.[40] The general tone of his complaints against Hobbes's procedures can be gathered from his remarks on article 18:

> What need is there for this whole apparatus you have, and for so many interspersed paralogisms and other suppositions which have nothing sound about them, in order to prove a thing so easy, and which no man would ever doubt? Unless perhaps someone had come across it in your demonstration, and seeing so many false and putrid things in it, he should assume that anything needing such things for proof must contain something wrong. (*Elenchus* 63)

Such a harsh tone is not altogether unwarranted, for Hobbes's work contains a considerable number of errors, both of omission and commission. The general intention of the chapter is clear enough and it has a sufficient connection to other mathematical work of the period to be reasonably intelligible, but it is hardly the magnum opus that Hobbes envisaged. In fact, the English version of this chapter differs from the Latin original, and many of the differences involve Hobbes unacknowledged incorporation of Wallis's criticisms.[41]

3.4.2 *Cavalieri and Hobbes*

Cavalieri's influence upon Hobbes's mathematical work is at least as strong as Galileo's, as we can see from chapter 17 of *De Corpore*.

40. Thus, in commenting upon the fourth article in chapter 16, Wallis remarks, "What should be put here in place of all of this you may learn from propositions 23, 69, and 71 of my *Arithmetica Infinitorum*. For what is treated universally in that place can easily be accommodated to this business" (*Elenchus* 46). Later, he suggests that a similar course of study would benefit Hobbes's presentation of parabolic curves: "But if you desire to understand this, you may learn it and things greater than it from propositions 23 and 64 of my *Arithmetica Infinitorum*" (*Elenchus* 51).

41. To take one example, Hobbes's original formulation of article 3 of chapter 16 (on the comparison of ratios of impetus in accelerated motion) was altered in the English translation to meet some of Wallis's objections, as Wallis himself points out in his response to the *Six Lessons*: "Now these two Objections [in the *Elenchus*] were clear and full, (and did destroy your whole demonstration;) and this you discerned well enough, though you did not think fit to make any reply or confession; (but invent some other objections, which I never made, that you might seem to answer to somewhat.) And therefore in the English, without making any words of it, you mend it" (*Due Correction* 97–98). Notwithstanding the changes, Wallis still found the English version of the article unacceptable, and even charged Hobbes with introducing more errors than were contained in the Latin original.

Hobbes was clearly familiar with Cavalieri's work from early on, for among his surviving papers is a notebook in which he copied extracts and summaries of Cavalieri's *Exercitationes Geometricae Sex*.[42] The notebook was probably used during Hobbes's extended stay in France in the 1640s—the period in which he assembled his *De Corpore* and when Cavendish enlisted his aid in Pell's campaign against Longomontanus.[43] Chapter 17 of *De Corpore* on the measure of "deficient figures" comes almost straight out of the *Exercitationes Geometricae Sex*, as we can see by comparing its second article with proposition 23 of part 4 of the *Exercitationes*. In Hobbes's parlance, the deficient figure *ABEFC* in figure 3.11 is produced by the motion of the line *AB* through *AC*, while *AB* diminishes to a point at *C*. The "complete figure" corresponding to the deficient figure is the rectangle *ABDC*, produced by the motion of *AB* through *AC* without diminishing. The complement of the deficient figure is *BDCFE*, the figure that, when added to the deficient figure, makes the complete figure. Hobbes's task is to find the ratio of the area of the deficient figure to its complement, given a specified rate of decrease of the quantity *AB*. He concludes that the ratio of the deficient figure to its complement is the same as the ratio between corresponding lines in the deficient figure and their counterparts in the complement. His statement of the theorem reads:

> A deficient figure, which is made by a quantity continually decreasing until it vanishes, according to ratios everywhere proportional and commensurable, is to its complement as the ratio of the whole altitude to an altitude diminished at any time is to the ratio of the whole quantity, which describes the figure, to the same quantity diminished in the same time. (*DCo* 3.17.2; *OL* 1:209)

42. Chatsworth House, Library, Hobbes MS. C.I.5.

43. The dating of the manuscript is not certain, but the hand and subject matter put it in the same period as MS. A.10 in the Hobbes archive at the Library of Chatsworth House. This manuscript is a draft of *De Corpore*, and it seems that Hobbes copied out the extracts from Cavalieri while he was writing *De Corpore*. The *Exercitationes* were published in 1647, and both Hobbes and Sir Charles Cavendish must have sought it eagerly. In a letter to Pell on 2/12 August 1648, Cavendish wrote that "Mr: Hobbes hath nowe leisure to studie and I hope we shall have his within a twelve-month." He then adds "I saw a booke at Paris of the excellent Cavalieros lately printed, concerning Indivisibles; whom you know was the Inventor or Restorer of that kinde of Geometrie; I had no time to reade it before I came awaye, and they are not to be bought; Mr: Careavin comming latelie from Italie brought this with him" (British Library Add. MSS. 4278, f. 273r). It is therefore no great stretch of the imagination to think that Hobbes was reading Cavalieri as he put together the mathematical sections of *De Corpore* in 1648 and 1649.

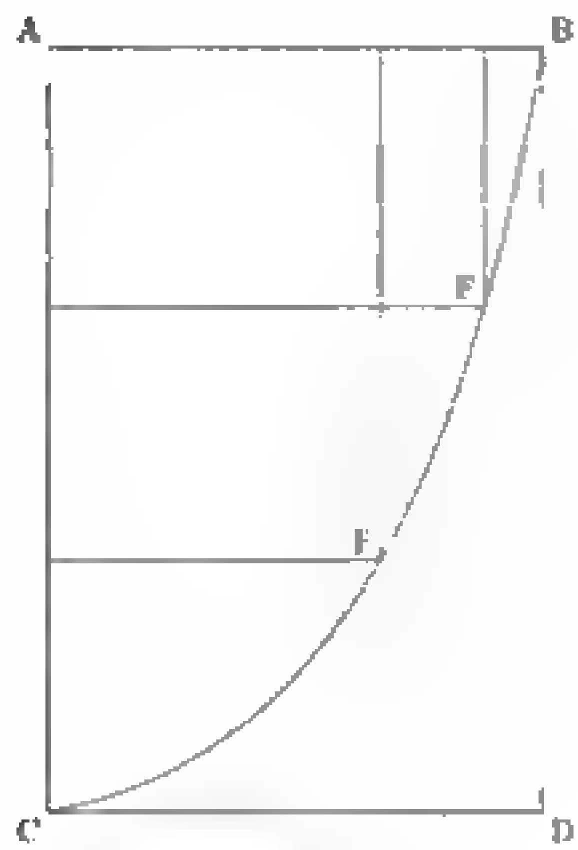

Figure 3.11

Thus, if the rate of diminution of *AB* is uniform, the line *ABEFC* will be a right line (indeed, the diagonal of the rectangle), and the deficient figure will be to its complement as one to one. In more complex cases, as when *AB* decreases as the square of the diminished altitude, the area of the deficient figure will be twice that of its complement. And, in general, if the line *AB* decreases as the power *n*, the ratio of the deficient figure to its complement will be n:1. Hobbes's proof procedure for this theorem involves the consideration of ratios between "all the lines" in the deficient figure and its complement, and shows a significant similarity to Cavalieri's famous procedure in the fourth of his six *Exercitationes Geometricae*—an exercise entitled *On the Use of Indivisibles in Cossic Powers.*[44]

There, Cavalieri pursued a result that historians of mathematics generally characterize as the attempt to prove the geometric equivalent of the theorem $\int_0^a x^n dx = a^{n+1}/(n+1)$. Except for differences in diagrams and terminology, Cavalieri's fourth *exercitatio* delivers the same results as Hobbes's account of deficient figures. In proposition 23 of *exercitatio* 4, Cavalieri asserts his version of the theorem we saw earlier from *De Corpore:*

44. Hobbes's proof in the original Latin version differs from that in the English version. In fact, the English version has two separate proofs, the second of which is a reworked version of the Latin original. Beyond this, the edition of *De Corpore* in Hobbes's 1668 *Opera* contains yet another variation on the same proof. Despite these differences, the proofs all depend upon concepts taken from Cavalieri. These various proofs are examined in section 2 of the appendix.

> In any parallelogram such as *BD* [as in figure 3.12], with the base *CD* as *regula*, if any parallel to *CD* such as *EF* is taken, and if the diameter *AC* is drawn, which cuts the line *EF* in *G*, then as *DA* is to *AF*, so *CD* or *EF* will be to *FG*. And let *AC* be called the first diagonal. And again as DA^2 is to AF^2, let *EF* be to *FH*, and let this be understood in all the parallels to *CD* so that all of these homologous lines *HF* terminate in the curve *AHC*. Similarly, as DA^3 is to AF^3, let also *EF* be to *FI*, and likewise in the remaining parallels, to describe the curve *CIA*. And as AD^4 is to AF^4, let *EF* be to *FL*, and likewise in the remaining parallels to describe the curve *CLA*. Which procedure can be supposed continued in the other cases. Then *CHA* is called the second diagonal, *CIA* the third diagonal, *CLA* the fourth diagonal, and so forth. Similarly the triangle *AGCD* is called the first diagonal space of the parallelogram, the trilinear figure *AHCD* is the second space, *AICD* the third, *ALCD* the fourth, and so on. I say therefore that the parallelogram *BD* is twice the first space, triple the second space, quadruple the third space, quintuple the fourth space, and so forth. (Cavalieri 1647, 279)

In its details Cavalieri's proof of this result departs from Hobbes's efforts. Cavalieri introduced the concept of "all the squares" of a figure, as well as "all the cubes," "all the square-squares," etc. He then argued that all the lines of a parallelogram are double all the lines of the triangle that is half the parallelogram, while all the squares of the same parallelogram are triple those of the triangle, and all the cubes of the parallelogram are quadruple all the cubes of the triangle. This, together with some other principles governing the comparison of lines in a fig-

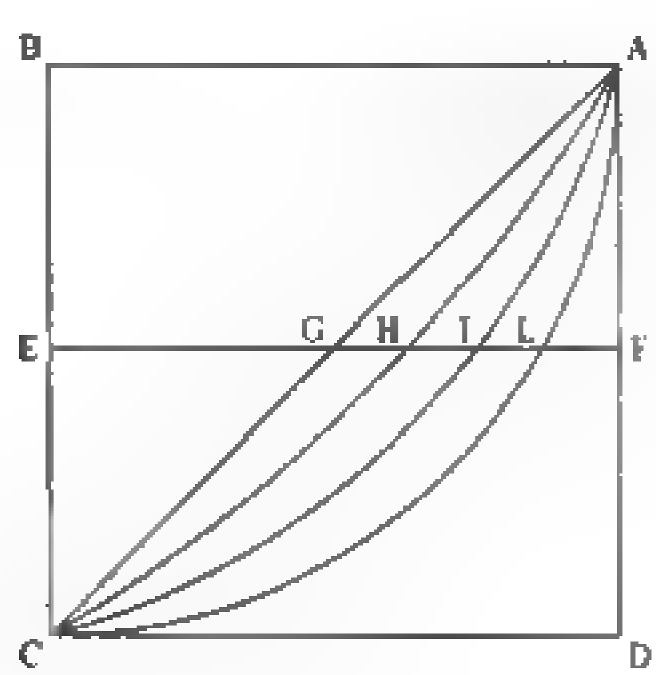

Figure 3.12

ure, yields Cavalieri's result.[45] Hobbes tried to establish the result by a more direct (if ultimately unsuccessful) argument that does not employ such devices as "all the squares" of a figure. These various attempts to prove this result are examined in the appendix and can be left aside here. Nevertheless, despite the differences in the details of the proofs, it is clear that Cavalieri's general approach was the model that Hobbes used in his treatment of deficient figures.

Such similarities between Hobbes and Cavalieri on this point are certainly striking, and it was obvious to Hobbes's contemporaries that this approach to quadratures had its roots in Cavalieri's work. In fact, Wallis eventually accused Hobbes of plagiarizing from Cavalieri, although when he wrote the *Elenchus* he could only voice his suspicion that much of chapter 17 of *De Corpore* must have been taken from someone else (*Elenchus* 83–84). In his *Due Correction* Wallis made the charge of plagiarism more specific by printing an excerpt from a letter alleging that "those propositions which Mr. Hobs had concerning the measure of the parabolasters were not his own, but borrowed from somebody else without acknowledging his author" and that "they were to be found demonstrated in an exercitation of Cavalerius *De Usu Indivisibilium in Potestatibus Cossicis*" (*Due Correction* 7).[46] Charges of plagiarism figure prominently in the dispute between Hobbes and Wallis, as we will see in more detail when we consider the question of Roberval's influence on Hobbes. For present purposes it is

45. The main points are argued in propositions 19–21 (Cavalieri 1647, 272–78). For an outline and analysis of the argumentation in the fourth *Exercitatio*, see Giusti 1980, 76–79.

46. The author of the letter is known to us only as a British gentleman by the name of Vaughan. In a letter to Hobbes of 19/29 December 1655 Stubbe reports, "Yesterday there came to mee one M:^{r} Vaughan of Jesus colledge desireing to know whether yo^{u} intended to answer Wallis. I assured him of it. whereupon hee told mee y^{t} y^{e} Gentleman whose letters in English d:^{r} W: had inserted was his brother & y^{e} occasion of y^{e} writeing y^{m} was this. When D:^{r} W's Elenchus came out, it was by him sent to his brother being a louer of y^{e} Mathematiques: who upon y^{e} perusall of it, sent him in a private letter y^{r} remarque, y^{t} D:^{r} W's had guessed aright, y^{t} such & such propositions were out of Caualllerius. M^{r} Vaughan being in company with Wallis, did let him know this. whereupon y^{e} D:^{r} upon y^{e} edition of his Arithmetica infinitorum, sent him one, together with a letter. The gentleman haueing receiued it, delayed an answer, y^{e} letter not requireing any; untill his brother did reminde him that y^{e} D:^{r} might expect a reply, and howeuer it would bee civill to returne him y^{e} complement. Upon this account hee wrote y^{e} letter you see, not intending y^{e} publication of it, or y^{t} euer hee should bee engaged in y^{e} defence of a booke hee had not then examined, when hee wrote y^{t}. hee is very much dissatisfyed with y^{e} publishing thereof, & would bee glad to see him corrected according to his demerit, & so would his brother too, he sayth" (*CTH* 1:394–95).

enough to note that there is sufficient evidence to make these charges quite plausible in the case of Hobbes and Cavalieri.

Aside from the obvious similarities in their methods, there are other important points of similarity between Hobbes's and Cavalieri's approaches to indivisibles. Both conceive of indivisibles as described by the motion of a line through a figure, and both rely upon the calculation of ratios between lines that is rooted in the Euclidean theory of proportions rather than the algebraic treatment of magnitudes characteristic of analytic geometry. This means that they do not attempt to represent the area of a figure as an infinite sum of infinitesimal elements. In contrast, Wallis does not appeal to motion, relies heavily upon arithmetical and algebraic methods, and does not use the painstaking comparison of ratios we find in Hobbes and Cavalieri. Most significantly, he is happy to analyze the area of a figure as an infinite sum, the terms of which are infinitely small in comparison with the total area. Some of these issues will reemerge in chapter 4 when we turn our attention to Hobbes's and Wallis's conceptions of the infinite.

3.4.3 Roberval, Hobbes, and the Rectification of the Spiral

The recognition of Hobbes's intellectual debt to Galileo and Cavalieri on these matters by no means rules out the possibility that his mathematical work was influenced by others. In particular, his knowledge of the method of indivisibles may also have come in part from his contact with Roberval in Paris during the 1640s. Roberval is an enigmatic character in the history of mathematics.[47] He held the chair in mathematics at the Collège Royal in Paris from 1634 until his death in 1675 and was a pioneer in the application of the method of indivisibles, but he actually published very little during his lifetime.[48] The mathematical chair at the Collège Royal was awarded on the basis of triennial competitions, and Roberval's reluctance to publish may well have derived from the fact that his continued livelihood could be ensured by keeping his methods and discoveries secret. Although he published very little, Roberval was nevertheless active in Parisian intellectual life; he was a

47. For more on Roberval see Auger 1962, Costabel and Martinet 1986, and Vita 1973. Walker 1932 is a dated but useful study of Roberval's principal work in the method of indivisibles.

48. Pierre Costabel observed that "[t]he discoveries that this mathematician and logician made, while being at once jealous of his priority and preoccupied with keeping his means secret, are the expression of a work much more important than that of the two books published in his lifetime" (Costabel and Martinet 1986, 23).

leading member of the "Mersenne circle," and several of his discoveries were first made public in some of Mersenne's publications.

Roberval's most important single work is the *Traité des Indivisibles,* which remained unpublished during his lifetime but appeared in 1693 as part of the *Divers Ouvrages de Mathématique et de Physique, par Messieurs de L'Academie Royale des Sciences* (Roberval 1693, 190–245). His procedures are very similar to those of Wallis, which is to say that Roberval considers plane surfaces as composed out of infinitely many indivisible parts, and the quadrature of a figure is typically effected by analyzing infinite sums of these indivisible elements. Hobbes's friend du Verdus was a pupil and close associate of Roberval, and he wrote a brief treatise outlining Roberval's methods that was later published as *Observations sur la composition de mouvements et le moyen de trouver les tangentes aux lignes courbes* in the *Divers Ouvrages* of 1693.[49] The *composition of motions* is Roberval's term for the method of analyzing a curve as the product of a point in motion, and particularly for the idea of considering the motion at any instant as a complex composition of two motions, such as Galileo's construction of the parabola from uniform rectilinear motion along one axis and uniformly accelerated motion along another.

It is the mention of Roberval in Mersenne's *Cogitata Physico-Mathematica* that led Wallis to include a specific allegation of plagiarism against Hobbes at the very end of the *Elenchus.*[50] Wallis reports that as he was finishing the *Elenchus,* and while much of it was already printed, he "unexpectedly came across some passages in Mersenne, which abundantly confirmed that which I have occasionally indicated earlier, namely that when something true is included among these things of yours, it is not really your own, but taken from somewhere else" (*Elenchus* 132). Earlier in the *Elenchus* Wallis had suggested the likelihood of Hobbes's plagiarizing those of his results that were actually sound, but he had been forced to conjecture about their possible source. However, his reading of the treatise *Hydraulica* contained in Mersenne's *Cogitata* led him to name Roberval as the source of one particular result concerning the arc length of the curve known as the Archimedean spiral.

The spiral of Archimedes is generated by the compound motion of a point that moves uniformly through a line that itself is rotating uni-

49. The treatise is in Roberval 1693, 67–111.

50. See Schumann 1995 for an overview of Hobbes's contributions to Mersenne's *Cogitata.*

formly about one of its endpoints. Thus, in figure 3.13, the spiral *abcdefn* arises from the motion of a point through the radius *an* in the same time that *an* completes a single revolution. The second corollary to the twenty-fifth proposition of the *Hydraulica* asserts that the first revolution of the spiral of Archimedes is equal to the arc length of a parabola having a base equal to the radius of the spiral and an axis equal to half its circumference. Mersenne reports that "when I was concerned with this result, a learned man proposed a certain right line that he thought equal to the first revolution of the spiral *abcdefn*, but the revolution of the spiral was greater than this proposed line, and our geometer showed that the spiral was equal to the parabola *GT*" (Mersenne 1644, 129). Roberval was always known by the epithet "our geometer" in Mersenne's text, and Wallis concluded that he must have been the source for Hobbes's result concerning the arc length of the spiral.[51] Elsewhere in Mersenne's *Cogitata*, Roberval is credited with other results with counterparts in *De Corpore*, and Wallis wasted no time in citing these as further instances of Hobbes's presumed plagiarism.[52]

Allegations of plagiarism were common fare in mathematical disputes of the seventeenth century, and there is nothing terribly remarkable about Wallis's use of such a charge against Hobbes. Roberval himself was quick to accuse others of the offense, and his allegations were made so frequently that Hobbes could remark that "Roberval has this peculiar trait: when someone makes public a great theorem he has found out, he immediately sends out letters in which he claims to have found the same result earlier" (*Examinatio* 5; *OL* 4:188). True to his form, Roberval circulated an open letter in which he accused both Hobbes and Wallis of appropriating various of his results, particularly those concerning the rectification of the spiral.[53] Wallis himself was

51. Hobbes's result and his proof are contained in the appendix.

52. Thus, Wallis claims that the results from chapters 17 and 23 of *De Corpore* concerning centers of gravity and the quadrature of parabolasters can be found in the treatise *Mechanica* among the *Cogitata*, while the method of composition of motions is contained in the *Ballistica*, and other results deriving from Descartes, Galileo, and Fermat are mentioned in the treatise *Mechanica* and in the *Reflexiones Physico-Mathematicae* at the end of the *Cogitata* (*Elenchus* 132–34).

53. The communication from Roberval (now lost) was brought by Thomas White to England, where Brouncker forwarded it to Wallis. The communication (which Wallis refers to as a "*charta*" in a letter to Brouncker that was intended for Roberval) was apparently a broadsheet accusing both Wallis and Hobbes of plagiarism. Wallis agrees that Hobbes is guilty of plagiarism and notes that he himself had attempted to contact Roberval and Gassendi (who had since died) to discern the source of Hobbes's mathematical results. However, he denies that he has ever seen Roberval's work and attempts

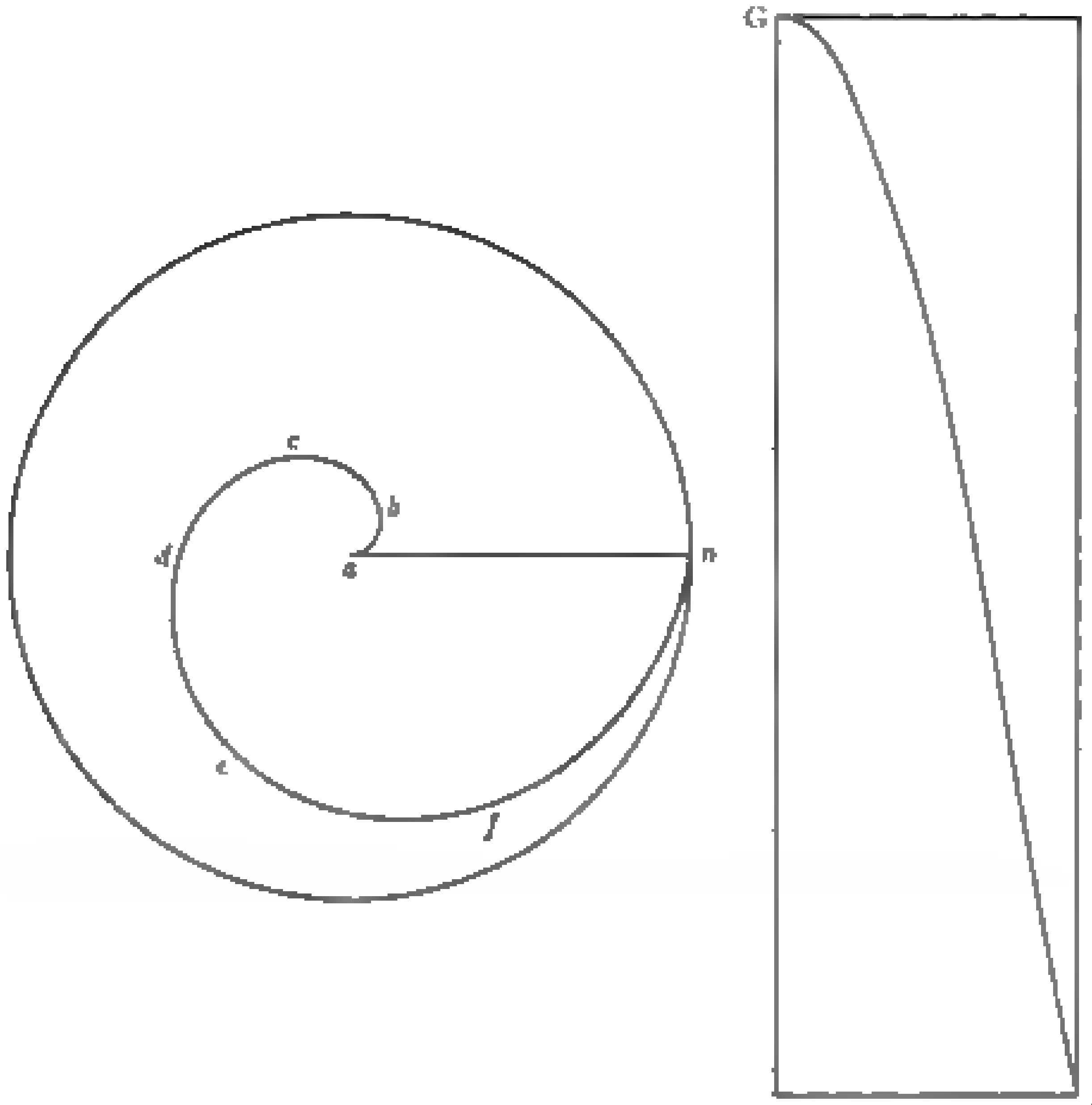

Figure 3.13

certainly no stranger to charges of plagiarism; he had a penchant for grinding self-promotion at the expense of others, and his career was consequently dogged by accusations that he had stolen most of the important work he published.[54]

Although these accusations of plagiarism are significant in the

to refute the charge that he is equally guilty of plagiarism. (Wallis to Brouncker for Roberval, 16/26 October 1656; Paris, Bibliothèque Nationale, nouv. acq. fr. 3252, f. 148–52). Wallis's letter to Gassendi requesting further information about the identity of the "vir doctus" mentioned in Mersenne's *Cogitata* is in Gassendi [1658] 1964, 6:540.

54. Wallis's proclivity for appropriating the work of others is detailed in a letter to Hobbes from Aubrey dated June 24/July 4 1675. In it, Aubrey gives an extremely unflattering portrait of a much-despised Wallis. He reports that Robert Hooke "has been as much abused by Dr Wallis as any one: he makes it his Trade to be a common-spye. steales from every ingeniose persons discourse, and prints it: viz from Sr Ch: Wren God knows how often, from Mr Hooke etc. he is a most ill-natured man, an egregious lyar and backbiter, a flatterer, and fawner on my Ld Brouncker & his Miss: that my Ld may keep up his reputation" (*CTH* 2:753). The letter goes on to detail similar charges against Wallis from other quarters. Even taking account of Aubrey's friendship with Hobbes and the latter's wish to hear Wallis denigrated, the letter suggests that Wallis was not a man to be trusted.

course of the Hobbes-Wallis controversy, it is difficult to conclude whether Hobbes actually owed much to Roberval. Because his published output was so small (at least as of the time of *De Corpore*), Roberval could not have had much of an influence on Hobbes by way of publications. The two were acquainted through Mersenne in the 1640s and must have discussed mathematical matters on many occasions (for instance, they both contributed to Pell's campaign against Longomontanus). Nevertheless, Roberval's well-known reticence makes it highly unlikely that he would have shared much with his English acquaintance. Hobbes's close friendship with du Verdus might have been the means of transmission for some of Roberval's ideas, since du Verdus was well versed in Roberval's methods and had prepared several summaries of his results. But there is no evidence that du Verdus met Hobbes before 1651 (which would be quite late in the preparation of *De Corpore*).[55] When pressed, Hobbes acknowledged that he had heard Roberval claim the rectification of the spiral, but denied that he was privy to the demonstration or that he had taken anything of substance from him.[56] In fact, Hobbes later insinuated that Roberval him-

55. Malcolm's account of du Verdus's life in the biographical register to Hobbes's *Correspondence* (*CTH* 2:902–13) is the best source for sorting out the details of the relationship between Hobbes and Roberval. If, as Malcolm suggests, Hobbes and du Verdus did not meet before 1651, this would place their first meeting well after the likely date of composition of a manuscript version of the first seventeen chapters of *De Corpore* (Chatsworth MS. A.10), which probably dates from the 1640s. Although the manuscript differs from the printed *De Corpore* in many respects, the seventeenth chapter (f. 26–30) does contain an analysis of deficient figures.

56. Recounting the matter in the *Six Lessons*, Hobbes declares,

> You say further (you the Geometrician) that I had the Proposition of the Spirall Line equall to a Parabolicall line from Mr. *Robervall*. True. And if I had remembered it, I would have taken also his demonstration, though if I had publisht his, I would have suppressed mine. I was comparing in my thoughts those two Lines, Spirall and Parabolicall, by the Motions wherewith they were described; and considering those Motions as uniform, and the Lines from the Center to the Circumference, not to be little Parallelograms, but little Sectors, I saw that to compound the true Motion of that Point which described the Spirall, I must have one Line equal to half the Perimeter, the other equal to half the Diameter. But of all this I had not one word written. But being with *Mersennus* and Mr. *Robervall* in the Cloister of the Convent, I drew a Figure on the wall, and Mr. *Robervall* perceiving the deduction I made, told me that since the Motions which make the Parabolicall Line, are one uniform, and the other accelerated, the Motions that make the Spirall must be so also; Which I presently acknowledged; and he the next day, from this very method, brought to *Mersennus* the demonstration of their equality. And this is the story mentioned by *Mersennus*, Prop. 25. Corol. 2. of his *Hydraulica*. (*SL* 6; *EW* 7:343)

self had first hit upon the solution to the question of the spiral only after Hobbes had put him on the right path.[57]

The problem of finding the arc length of the spiral acquired an especially significant status in the dispute because Wallis himself fell into error in his attempts to solve it. Moreover, Wallis made what is probably the same error as Hobbes had made when he first proposed the rectification of the spiral to Mersenne and Roberval. The difficulty can be summarized as follows, using the techniques of calculus to compute the arc length of the spiral.[58] Let r designate the distance moved along the radius R, s the length of the spiral at r, and φ the angle of rotation at r (see figure 3.14); then (using polar coordinates) we get $r = R\varphi/2\pi$, and $ds = \sqrt{dr^2 + r^2 d\varphi^2}$. Using the general arc length formula $s = \int_0^{2\pi}\sqrt{r^2 + (dr/d\varphi^2)}d\varphi$, the arc length will then be an integral of the form $\int_0^{2\pi}ds$, from which we obtain the result $s = R/2\pi\int_0^{2\pi}\sqrt{1 + \varphi^2}d\varphi$. In his *Arithmetica Infinitorum,* Wallis mistook the problem as one of computing the aggregate of arcs of the spiral's limiting circles, rather than the aggregate of the elements of the spiral. In other words, he computed the integral $\int_0^{2\pi}rd\varphi$, which easily solves to πR. Wallis thereby obtained the false result that the length of the spiral is one-half the circumference of the circumscribing circle. The source of the confu-

57. In the fifth dialogue of his *Examinatio,* Hobbes tells essentially the same story about the origin of his rectification of the spiral as he had reported in the *Six Lessons;* however, he adds that there was a fourth person present when the problem was first discussed, and this witness was supposedly prepared to give Hobbes credit for the discovery. An October 1656 letter from Claude Mylon to Hobbes is the basis for this version of the story. Mylon wrote, "M. de Carcavi told me that M. Roberval had demonstrated geometrically the equality between a spiral and a parabola, a proposition which on a previous occasion he had demonstrated by the motion of a point. As you know, you gave him the notion of finding it" (*CTH* 1:315). Writing in the third person, Hobbes declares in the *Examinatio* that "Hobbes sought by letter from the fourth person, whom he did not name, whether or not this sheet [claiming Hobbes and Wallis had plagiarized from Roberval] was Roberval's. He responded that he did not know whose it was, but that he was ready to testify that the inspiration *[lucem]* and the method of demonstration had been taken from Hobbes by Roberval" (*Examinatio* 5; *OL* 4:190). Although he has embellished the account somewhat, Hobbes is clearly referring to Mylon's letter here.

58. The problems involved in this case are by no means trivial, and it seems that Roberval spent a good deal of time unraveling them. See Auger 1962, 72–74, for a summary of the problem and Pedersen 1970, which reproduces two manuscripts containing Roberval's solution, for a more detailed account of the matter. Breidert 1979, 418–20, deals with essentially the same issue, and also in the context of the Hobbes-Wallis dispute.

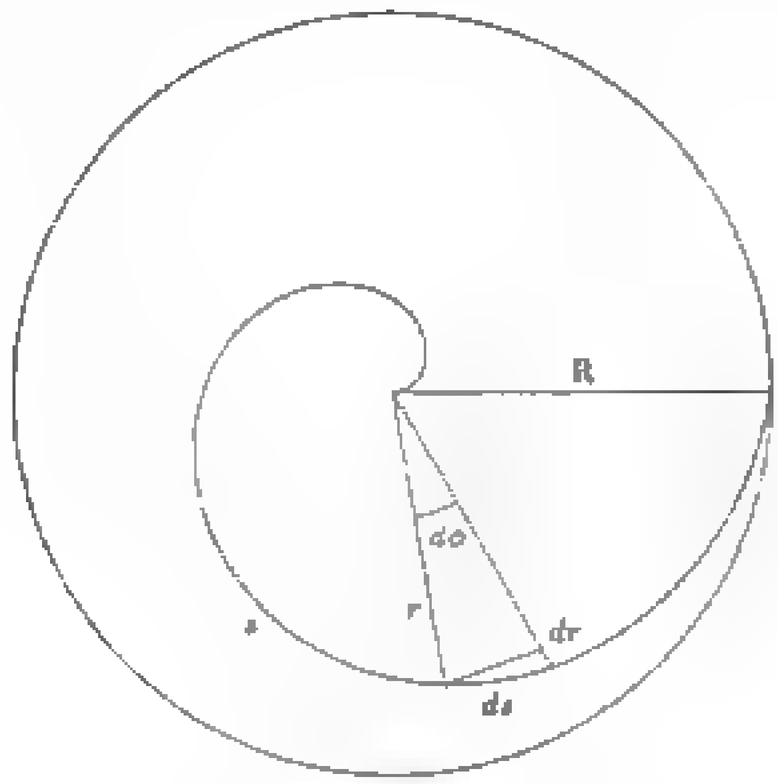

Figure 3.14

sion, as D. T. Whiteside has remarked, is that Wallis's procedure mistakes two curve transformations, one of which is length preserving while the other is area preserving.[59]

When Mersenne reported that the "learned man" Hobbes had erred in solving the problem and proposed a line greater than the true result, he presumably was referring to this very kind of error. Hobbes was apprised of the nature of his mistake by Roberval, who pointed out that the spiral is produced by two motions—one uniform and the other accelerated—while the line whose length Hobbes computed is traced by two uniform motions. Although uniform *circular* motion is included in the definition of the spiral, the point that traces the spiral also passes through the radius of the circular motion; consequently, it lies in the periphery of ever greater circles, which increase with the square of the time. Roberval reasoned that the resulting composition of motions yields the parabola as the arc length of the spiral. In any event, proposition 5 of Wallis's *Arithmetica Infinitorum* originally declared that the length of the Archimedean spiral is one-half the circumference of its limiting circle. It was apparently only after he had seen Roberval's solution to the problem (as reported in Mersenne's *Hydraulica*) that Wallis realized he had made an error, and he hastily attempted to correct it by adding a scholium to the thirteenth proposition that declared that his expression "spiral line" was not intended to

59. See Whiteside's note to a Newtonian treatise on fluxions (Newton 1967–81, 3:308 n. 704).

be the spiral of Archimedes, but rather a collection of arcs of the circles that are circumscribed about the true spiral.[60]

Hobbes jumped at the chance to expose this error, and insisted that "the Aggregate of that infinite Number of infinitely little Arches, is not the Spirall line made by your construction, seeing by your construction the Line you make is manifestly the Spiral of Archimedes; whereas no Number (though infinite) of Arches of Circles (how little soever) is any kind of Spirall at all; and though you call it a Spirall, that is but a patch to cover your fault, and deceiveth no man but yourself" (*SL* 5; *EW* 7:311). In point of fact, when Wallis critiqued Hobbes's comparison of the Archimedean spiral with the parabola (*Elenchus* 125–26) he plainly showed that he had not properly understood the construction of the spiral or its proper rectification. He there faults Hobbes for taking the spiral to be compounded of a uniform and an accelerated motion, and characterizes the Archimedean spiral as a curve generated by the concourse of two uniform motions, writing, "If you had understood the generation of both [the spiral and the parabola], you would instead have concluded that their generation is dissimilar, because the one is generated from the concourse of two motions, of which one is uniform and the other uniformly accelerated, while the other is generated from the concourse of two motions both of which are uniform" (*Elenchus* 126). Hobbes's reply correctly points out that because the point describing the spiral "goes on uniformly in the Semidiameter, it

60. Wallis declares, "It is to be noted that in the preceding propositions concerning the spiral line (some of which I will use in what follows), I have used the term 'spiral line' rather loosely in order to avoid many cumbersome locutions. That is, by the term 'spiral line' (whenever it is compared to the periphery), I wish to be understood (as I indicated in proposition 5) 'the aggregate of all the infinite number of arcs of similar sectors, from which is composed that figure in which the infinite number of sectors of the spiral are inscribed' (which is the meaning I have used in proposition 5 of this work and which Archimedes used in proposition 21 etc. of his treatise on spirals). This spiral taken in the proper sense is always less than this aggregate, most of all near the beginning of the spiral" (1657c, 3:10–11).

The fifth proposition of the *Arithmetica Infinitorum* (which begins the treatment of the spiral and its rectification) was altered in the first volume of Wallis's *Opera Mathematica,* which appeared in 1695. There, he adds a note to the beginning of the proposition, explaining that the spiral he is about to consider is not that of Archimedes (*OM* 1:397). Hobbes's reply to Wallis's original (1656) attempt to mask his error is interesting: "your spiral puts me in mind of what you have underwritten to the diagramme of your prop 5. *The spirall in both figures was to be continued whole to the middle, but by the carelessnesse of the Graver it in one figure* manca, *in the other* intercisa. Truly Sir, you would hardly make your Reader beleeve that a Graver could commit those faults without the help of your own Coppy, nor that it had been in your coppy, if you had known how to describe a spiral line then as now" (Στίγμαι 1; *EW* 7:372).

is impossible it should not pass into greater and greater Circles, proportionally to the Times; and consequently it must have a swifter and swifter Motion circular, to be compounded with the uniform Motion in every Point of the Radius as it turneth about" (*SL* 5; *EW* 7:327).

This episode serves to indicate several important features of the mathematical part of the controversy between Hobbes and Wallis. First, although Hobbes's mathematical work in *De Corpore* is beset with numerous technical and conceptual errors, he pursued problems familiar to the mathematical world of his day and his methods are not completely foreign or unrecognizable. Second, although Wallis is frequently justified in his harsh criticisms, he could clearly overplay his hand in attempting to destroy the entire edifice of Hobbesian mathematics. In particular, Wallis's original misunderstanding of the Archimedean spiral and his clumsy efforts to cover his tracks (particularly in his reply to Roberval's charges of plagiarism) show that Hobbes was not alone in his capacity for mathematical error.[61]

3.5 CHAPTER 20 AND THE FAILED CIRCLE QUADRATURE

Hobbes intended the centerpiece of *De Corpore* to be his quadrature of the circle in the twentieth chapter. This is not to say that he regarded the remainder of his treatise as unworthy of praise or even adulation, but it is clear that he hoped to assert preeminence in the learned world largely on the basis of the solution of the problem of squaring the circle. This intention was not, however, part of the original plan for the work, because the twentieth chapter is a late insertion and not part of the work as originally planned.[62] Indeed, the lack of apparent con-

61. This is quite clear in Wallis's letter to Roberval (via Brouncker), which contains an extended but futile argument to show that the passages in the *Elenchus* do not betray a misunderstanding of the case, but instead take Hobbes's spiral to be the same kind as that treated in propositions 5–13 of the *Arithmetica Infinitorum* (Wallis to Brouncker for Roberval, 16/26 October 1656; Bibliothèque Nationale, Paris, nouv. acq. fr. 3252, f. 148–52).

62. It is noteworthy, for example, that the surviving manuscripts of *De Corpore* do not contain the circle quadrature, although they closely follow the printed version in many other respects. Chatsworth MS. A.10 contains drafts of eighteen chapters grouped into two parts. The first part consists of six chapters that correspond very closely to those printed in *De Corpore*. The second part of MS. A.10 consists of twelve chapters, the first six of which closely parallel the structure and wording of the printed *De Corpore*. There is nothing in this manuscript to correspond to the printed twentieth chapter, although the remaining six manuscript chapters deal with a variety of mathematical matters. Moreover, Chatsworth MS. A4 is a fine copy of a nineteenth chapter dealing

nection between Hobbes's twentieth chapter and the surrounding material led Wallis to complain that chapters 18–20 "hang together *like a rope of sand*," because "they have no connexion at all" (*Due Correction* 126). As we have seen, in the *Vindicae Academiarum* Ward challenged Hobbes to make public his geometrical discoveries, and this provoked him to include an attempt at circle quadrature in *De Corpore*. Ward's obvious lack of respect for his mathematical abilities must have enraged the sage of Malmesbury, for he included in chapter 20 several petulant outbursts directed at "Vindex"—his term for the author of the *Vindicae Academiarum*. Thus, in the appendix to the *Vindicae Academiarum* Ward reported that he had "heard that Mr. *Hobbs* hath given out, that he hath found the solution of some Problemes, amounting to no lesse than the Quadrature of the Circle" and promised to "fall in with those that speake loudest in his praise" if Hobbes would publish a successful quadrature ([Ward] 1654, 57).

Confident that he had squared the circle and plainly relishing the chance to humble Ward, Hobbes addressed him contemptuously in the first impression of *De Corpore* with the words:

> But you who are the author of that defamatory book entitled *Vindicae Academiarum*, whatever your concealed name may be, you inept defender [*Vindex*] of the academies which I have never injured; my lying accuser: I absolve you from the payment of those promised praises due to me for solving this problem, judging that to be praised by a man of your manners is considerably less noble than to be censured.[63]

with the description and measure of parabolic arcs and spirals; this differs significantly from the published chapter 19 (which deals with angles of incidence and reflection), although some of the material made its way into the published chapters 17 and 18. Furthermore, Sir Charles Cavendish's notes on an early manuscript version of *De Corpore* make no reference to a circle quadrature, although many mathematical topics are discussed. These manuscript versions probably date from the 1640s, and it is certainly possible that Hobbes decided to include a circle quadrature in *De Corpore* at that time, or in the decade or so before the actual appearance of the finished work. Nevertheless, the absence of any quadrature in the manuscript and the fact that Hobbes was drafting and redrafting the twentieth chapter even as *De Corpore* went to press both suggest that it was added hastily. On Sir Charles Cavendish, his connection with Hobbes, and the writing history of *De Corpore*, see Jacquot 1949, 1952a, 1952b, and 1952c.

63. This passage was part of the first impression of *De Corpore*, but the type was reset to tone it down somewhat. The original Latin survives only in Wallis's *Elenchus* at page 121. As first set, the passage read: "Tu vero libelli famosi, qui titulus Vindiciae Academiarum, quicunque celato nomine author es, Academiarum quas nunquam laesi inepte Vindex, Accusator mei mendax, promissas mihi ob soluta haec problemata laudes

Unfortunately for Hobbes, his triumphal outburst was (to say the least) premature. While *De Corpore* was still in press, he showed his attempted quadrature to some friends, who advised him of his error and convinced him to revise the twentieth chapter of his magnum opus. Hobbes reworked the failed quadrature, downgrading it from what had been advertised as the problem "To find a right line equal to the perimeter of a circle," and retitling it "From a false hypothesis, a false quadrature." But he left the construction standing and added a brief demonstration of its falsity. In the published version of the first article of chapter 20, Hobbes admits his error and adds an interesting admission when he comments that he was prepared to declare the problem solved by his initial attempt "except that the insult of malevolent men held me back from this overly hasty conclusion, and made me examine the matter more closely with friends" (Hobbes 1655, 171). It is unclear who these "malevolent men" may have been, and it is also unclear who advised Hobbes of his errors, but it is evident that Hobbes was eager to humble his opponents by making his mathematical powers manifest.[64]

Still convinced that the problem of quadrature could be solved by means no more complex than the simple constructions he had employed earlier, Hobbes attempted another quadrature, but this was ultimately downgraded to a mere approximation. Rather than showing how to find a right line equal to the quadrant of a circle (which, as we have seen, is equivalent to the quadrature of the circle) the second article of chapter 20 was reset to announce only "the finding of a right line as nearly equal to the quadrant of a circle as desired" (Hobbes 1655, 170). This is hardly the magnificent result that Hobbes had in-

istas tibi condono: laudari ab homine tuis moribus quam vituperari aliquantò minùs honestem esse judicians." As reset, it reads: "Scio Authorem libelli famosi cui titulus Vindiciae Academiarum, quicunque celato nomine is fuerit, Academiarum quas nunquam laesi ineptum Vindicem, accusatorem meum mendacem, promissas mihi laudes non praestiturum, sed dicturum non deberi. Libenter autem illis careo, vel si debeantur condono laudari ab homine illius moribus quam vituperari aliquanto minùs honestum esse judicans" (Hobbes 1655, 174).

64. It is interesting to note that even as good a friend as du Verdus found Hobbes's decision to publish the "false quadrature from a false hypothesis" rather odd. In a letter to Hobbes of December 1655, he observes that "I might also suspect your motive for putting the section 'A false squaring of the circle from a false hypothesis' in ch. XX, were it not the philosopher's job to give examples of false reasoning just as much as of true reasoning. Besides, it is the action of a generous spirit to admit freely that he has held beliefs which he later learned were false" (*CTH* 1:226). Schumann (1997, xlv n. 65) suggests William Brereton (a friend of Hobbes and student of Pell) as the person who convinced Hobbes of the error in his first attempts at quadrature.

tended, and (as Wallis was happy to point out) it went no further than results that were already well known.[65]

The third quadrature was put forth as an exact result that follows from a method for the arbitrary section of angles. Of course, it too is a failure, and Hobbes admitted that he was made aware of the error only after *De Corpore* was well through the printing process. He was thus forced to add a brief paragraph to the end of chapter 20, declaring:

> Since (after it was written) I have come to think that there are some things that could be objected against this quadrature, it seems better to warn the reader of this than to delay the edition any further. It also seemed proper *[placuit]* to let stand those things that are deservedly directed at Vindex. But the reader should take those things that are said to be found exactly of the dimension of the circle and of angles as instead said problematically. (Hobbes 1655, 181)

Naturally, the net effect of these three failed quadratures is to undermine any claim Hobbes might make for the superiority of his mathematical methodology, at least insofar as such a claim is founded upon his success in the solution of the great classical problems.

Sensitive to the damage that the false quadratures must inevitably do to his reputation, Hobbes attempted to salvage the matter by toning down his harsh words addressed to Ward, and reset his polemic against Vindex to read in the third person:

> I know that the author of that defamatory book entitled *Vindicae Academiarum* (whatever his concealed name may be), that inept defender of the academies which I have never injured, and my lying accuser, will not offer those promised praises to me, but will say they are not to be given. But I freely go without them, or

65. Hobbes declares that "geometers can so determine the perimeter of the circle in numbers as to differ from the true value by less than any desired error. Let us then see whether the same can be done by the drawing of right lines" (Hobbes 1655, 172). Wallis responds that "[i]t can most assuredly be done. That is, it is required that there be right lines proportional to these discovered numbers. Or also, that the arithmetical operations by which these numbers are arrived at are carried out in lines. And then whatever can be done in such numbers, the same can be effected in lines, if the same operations are carried out in lines (which is easily done). And thus, if this is what you have determined to do, you will show nothing great, for it is now agreed that this can be done in many ways" (*Elenchus* 119). In fact, Hobbes's construction here adds nothing to what had been shown by Archimedes, Huygens, and Snell. We will see examples of their approximation techniques of the value of π in chapter 6.

> if they are due I absolve him from the payment of them, judging that to be praised by a man of his manners is considerably less noble than to be censured. (Hobbes 1655, 174)

He retained, however, another retort to Ward at the end of the third quadrature, in which he announced that "[y]our praises (Vindex) are now owed to me, except that I do not desire them" (Hobbes 1655, 176). But in reality, even Hobbes's more modest estimates of his mathematical success proved vastly overstated and would come back to haunt him.

Making matters much worse for Hobbes was the fact that Wallis had procured an early unbound copy of the first impression *De Corpore,* and was able to reconstruct the unflattering history of Hobbes's attempts to square the circle. This gave Wallis the opportunity to recapitulate Hobbes's earlier efforts and cast an extremely unwelcome light upon the state of his mathematical knowledge as well as his judgment. Much of Wallis's program in the *Elenchus* is devoted to precisely this task, which gave him the chance to taunt Hobbes with the observation that "after privately and publicly, here and abroad, you had boasted that you would bring forth other unknown marvels beyond the quadrature of the circle (which was such a small thing with you); after first, second, and third revisions, you print it a first, second, and third time: in the end you dexterously admit that you have by no means brought forth what you had promised; and then you ask that these things be taken 'as said problematically'" (*Elenchus* 2–3).

Hobbes's disappointment at his failed quadratures was made all the more bitter by the fact that Wallis so publicly paraded the sorry history of chapter 20 of *De Corpore.* Having been provoked by Ward into printing a hastily assembled quadrature and then replacing it with others of no better quality, Hobbes was forced to undergo the humiliation of seeing his efforts dissected and the many layers of error peeled away one by one.[66] So public a failure of his mathematical project would

66. Robertson aptly summarized the episode as follows:

> One of some copies carelessly issued in the first unamended form having fallen into his hands, [Wallis] was not the man to scorn such a weapon of ridicule, and from it, with his unbound copy, he was able to spell out the whole history of Hobbes's doings from the time of Vindex's challenge. This accordingly he laid bare, showing how, shaken from a brief illusion of triumph by friends concerned for his reputation, again and again during the year had Hobbes persisted in printing loose assumptions as strict truths, and rough and contradictory approximations as exact solutions of an impossible problem; till at last, rather than delay his book longer, he was fain to be content with his lame and impotent conclu-

have discouraged a man of fainter heart than Hobbes. But instead of admitting defeat, Hobbes resolved to vindicate his mathematics and overthrow the entire mathematical corpus of Wallis; the result, as we have already outlined in chapter 1, was that from 1655 until the very end of his life, Hobbes was engaged in his war with Wallis.

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sions, made grotesque by the side of the jubilant bursts which he had not the heart to suppress, because he had once had the joy of giving vent to them. (Robertson [1886] 1910, 173–74).

CHAPTER FOUR

Disputed Foundations

Hobbes vs. Wallis on the Philosophy of Mathematics

> It is well for you that they who have the disposing of the professors places take not upon them to be Judges of Geometry. For if they did, seeing you confesse you have read these Doctrines in your School, you had been in danger of being put out of your place.
>
> —Hobbes, *Στίγμαι*

Wallis's uncompromising critique of Hobbes was not confined to a purely technical exposition of flaws in his attempted quadratures. He objected to the entire Hobbesian program for mathematics, and what little he did not denounce as ill founded he dismissed as unoriginal, if not plagiarized. Hobbes's efforts in defense of the mathematics of *De Corpore* led him to oppose nearly every point raised by Wallis and, following the seventeenth-century conventions of polemical literature, the dispute between them became an exchange of point-by-point rebuttals of the opponent's latest piece, intermixed with the occasional new charge of folly or ineptitude. Aside from the dubious virtues of sheer pedantry, it makes no sense to go through all of these charges and countercharges. However, the two men also debated significant philosophical problems in mathematics, and the object of this chapter is to outline the most salient of such debates. Again, the disputants' practice of trading rebuttals over previously covered points means that it is frequently not worthwhile to follow an exchange on a particular point in complete detail over more than two decades, so I will refrain from giving an exact chronology of the charges and countercharges. The mathematical themes I find most worth exploring include the role of physical concepts in the philosophy of mathematics, the nature of ratios, the problem of the "angle of contact" between curve and tangent, and the admissibility of infinitesimal methods. This

chapter is concerned with outlining these contested points and examining their role in the Hobbes-Wallis controversy. My account requires some excursions into mathematical and historical material that is not often covered, but an adequate understanding of the contested issues requires that we take some relative arcana into consideration. I hope ultimately to show that Hobbes and Wallis adhered to widely divergent traditions in mathematics and that they held completely different conceptions of what mathematics is and how it should be developed.

4.1 MATHEMATICS, PHYSICS, AND THE NATURE OF BODY

Wallis condemned Hobbes's attempt to place mathematics on a thoroughly materialistic foundation, and this aspect of their dispute evolved into a lengthy exchange over the relationship between mathematics and natural philosophy. Wallis was by no means unique in objecting to Hobbes's materialism. As we saw in chapter 2, critiques of materialism were common fare among the many anti-Hobbes polemics in seventeenth-century England, but these tended to focus on the theological and moral dangers alleged to follow from the denial of immaterial substances. Hobbes's critics argued that the rejection of incorporeal substances made it impossible to accept the Christian account of an afterlife, that it reduced God to the status of a material being, and implied that humans have a merely animal (or even mechanical) nature, devoid of moral value or obligations. Ward's *Philosophicall Essay* of 1652 attacked Hobbes's materialism because it threatened the doctrine of immortality, but this was not the end of Ward's campaign against Hobbes; he returns again and again in his 1656 *Exercitatio* to the theme that the denial of immaterial substances is an unsupported dogma inimical to theology and morality.[1] Wallis certainly shared the widespread repugnance over Hobbes's materialism, but his published dispute with Hobbes concentrated on the mathematical issues connected to the Hobbesian conception of body and the foundational role of material concepts in his philosophy of mathematics.

1. Thus, Ward insists, "It should first have been shown that there is nothing else contained in the universe except bodies and their motions, and that on that account God, angels, and the eternal souls of men are to be done away with. For this is what he asserts everywhere in *Leviathan*, that there is no spirit, but that it is an absurd connection of words 'spiritual substance' which philosophers are accustomed to have in their ears" (*Exercitatio* 21).

Wallis opposed Hobbes's materialistic foundation for mathematics on two related grounds: first that it fails to accommodate the abstract, immaterial character of pure mathematics, and second that it employs physical principles that are at least doubtful, if not actually false. Although these two objections are clearly related, it is worthwhile to treat them separately because doing so can help illuminate Hobbes's conception of physics and its relationship to mathematics.

The charge that Hobbes degrades mathematics by founding it on the principles of body is an immediate consequence of Wallis's adherence to a traditional distinction between pure and applied mathematics.[2] According to the tradition, pure mathematics is distinguished from its applied counterpart by the fact that it deals with abstract objects not found in nature. Instead, these abstractions inhabit the (presumably immaterial) realm of the intellect, and the theorems of mathematics are thus freed from dependence upon features of the material world. As we saw in chapter 3, Hobbes largely ignored this classical division by developing an account of quantity that makes physical body the sole subject of quantity.

The traditional account of the relationship between mathematical and physical principles derives much of its plausibility from the idea that (purely) mathematical facts do not—indeed must not—depend upon the structure or contents of the material world. Part of what gives mathematics its particularly exalted epistemological status is the intuition that mathematical theorems would remain true even if the world were vastly different than it actually is. We are all inclined to accept that even if there were no bodies at all, or even if the nature of matter were entirely different, 17 would still be a prime number and the Pythagorean theorem would still hold. The traditional distinction between pure and applied mathematics accommodates this idea by locat-

2. Although Wallis accepts a traditional division between pure and applied mathematics, he departs from the tradition by claiming that arithmetic is a more abstract and pure science than geometry: "Therefore we say that there are only two pure mathematical disciplines, namely geometry and arithmetic. The first of these deals with discrete quantity or number, the other with continuous quantity or magnitude. But of these indeed one is more and the other less pure: for the subject of arithmetic is so to speak purer and more abstract than the subject of geometry, and on that account it contains more universal speculations that are equally applicable to geometrical matters and to others" (*MU* 1; *OM* 1:18). Nevertheless, he is firmly committed to the view that abstract mathematics deals with objects not found in nature. Moreover, he insists that the truths of mathematics cannot be concerned with the nature of body. Indeed, it is apparently geometry's connection with the nature of space (and hence body) that renders it less pure and abstract in Wallis's estimation.

ing mathematical truth in the relations among abstract objects whose properties are independent of the structure or contents of the actual world. An application of mathematics requires, on this view, additional nonmathematical hypotheses that link the necessary truths of mathematics to contingent features of the physical world.

In his *Elenchus*, Wallis repeatedly castigates Hobbes for introducing physical (rather than mathematical) principles into his account of quantity and thereby contaminating the foundations of mathematics. He asks, for example, "[W]hat need is there for the concepts of body or motion [in Hobbes's definitions of point and line], since the concept of a line can be understood without them?" (*Elenchus* 6). He explains that such physical notions are "plainly accidental, nor do they pertain to their essences, so it is strange to find motion in the definition of a point or line" (*Elenchus* 7). He objects for similar reasons to Hobbes's fundamental dictum that "a body always retains one and the same magnitude, when at rest or moved, but when moved it does not retain the same place" (*DCo* 2.8.5; *OL* 1:93). Wallis holds that this principle is "plainly physical, nor does it regard the present matter, nor is it in any way connected with it, so to what purpose was any mention to be made of it in this mathematical definition of yours?" (*Elenchus* 10). Hobbes's definition of equal ratios in terms of bodies producing equal effects in equal times (*DCo* 2.13.6; *OL* 1:132) meets the same objection. Wallis demands what relevance the definition has to the nature of mathematics and asks "to what end, in order that this notion be understood, need there be a consideration of time or weight, or of any other quantity?" (*Elenchus* 19). In these and other cases Wallis charges that Hobbes takes simple and clear concepts from pure mathematics and renders them complex and obscure by defining them in terms inappropriately borrowed from natural philosophy.

Hobbes's response to such criticisms is to insist that it is only by the introduction of physical concepts that mathematics can be grounded on a firm foundation. Indeed, he holds that "[f]or men that pretend no less to naturall Phylosophy, then to Geometry, to find fault with bringing Motion and Time into a Definition, when there is no effect in nature, which is not produced in Time by Motion, is a shame" (*SL* 3; *EW* 7:242). Thus, where Wallis finds the recourse to time, motion, or body to be the intrusion of nonmathematical concepts into the foundations of mathematics, Hobbes retorts that no genuine demonstration is possible unless it proceeds from the causal principles of body and motion. This aspect of his philosophy of mathematics is

summed up in Hobbes's 1666 essay *De Principiis et Ratiocinatione Geometrarum:*

> "But," you will ask, "what need is there for demonstrations of purely geometric theorems to appeal to motion?" I respond: First, all demonstrations are flawed, unless they are scientific, and unless they proceed from causes, they are not scientific. Second, demonstrations are flawed unless their conclusions are demonstrated by construction, that is, by description of figures, that is, by the drawing of lines. For every drawing of a line is motion: and so every demonstration is flawed, whose first principles are not contained in the definitions of motions by which figures are described. (*PRG* 12; *OL* 4:421)

This part of the controversy thus reduces to a conflict over the proper demarcation between mathematics and natural philosophy. As such, it is not readily amenable to resolution, because such fundamental disagreements can rarely be overcome or even profitably debated. Hobbes continued to insist that the only proper foundation for mathematics must be sought in the nature of body, while Wallis viewed this program as an abandonment of the essential features of mathematics. This aspect of the dispute can be illustrated in the exchange drawn from Wallis's *Due Correction*, where he writes: "You adde, *That there is no man (beside such Egregious Geometricians as we are) that inquires the Equality of two bodies, but by measure: And, as for liquid bodies; &c. by putting them one after another into the same vessel, that is to say, into the same place; And, as for hard bodies, they inquire their Equality by weight:* To which I shall reply nothing at all; because you speak therein so like a Geometrician" (*Due Correction* 55–56). Wallis obviously attempts irony in his remark that Hobbes "speaks so like a Geometrician," and his refusal to reply shows that he takes this treatment of measure to be far removed from anything in a genuine science of geometry.

Wallis further faulted Hobbes's materialistic philosophy of mathematics for its dependence upon controversial physical principles, and in particular upon the principle that a body always retains the same quantity. Without declaring his own opinion in the matter, Wallis observed that this tenet had been denied by those who hold that the same body can have lesser or greater quantity as a result of the contrary processes of rarefaction and condensation. In defining equal bodies as those that can occupy the same place (*DCo* 2.8.13; *OL* 1:99–100),

Hobbes identifies the magnitude or quantity of a body with the place it can occupy. But Wallis objects to the definition on the grounds that it "proceeds from the supposition of things that are not true, or that if true must be proved and not supposed" (*Elenchus* 9). In particular, he claims that

> [w]hen you define equality or inequality of one body to another, not by the place it occupies, but by that which it can occupy; I ask what happens if the same body could now occupy a greater and now a lesser space, as for example by rarefaction or condensation? Indeed, if it is conceded that this could be, then the same body could be with respect to another simultaneously equal, greater, and less (which is so absurd that even you must see it to be so); or indeed it could be that (whatever place it may now actually occupy) whenever it then moves a body can occupy the same, as well as a greater or lesser space (successively, just as is necessary in order that your definition be understood). (*Elenchus* 9)

The point at issue here is not whether rarefaction and condensation are processes that actually take place in nature, but whether it is conceivable that there could be such processes. Wallis finds it absurd that Hobbes should adopt an account of mathematics that rules out rarefaction and condensation as necessarily false doctrines, since this makes the science of mathematics (a realm of necessary truths independent of physical matters) contingent upon the status of disputed physical principles.

The theory of rarefaction and condensation has its roots in book 4 of Aristotle's *Physics*, but it was not confined to classical Greek sources. Several seventeenth-century natural philosophers endorsed the theory, among them Sir Kenelm Digby. Digby's natural philosophy was an idiosyncratic attempt to combine the teachings of Aristotle with the mechanistic "new philosophy" of Galileo, Descartes, and other notable modern authors. His *Two Treatises* of 1644 examined the nature of body and the soul, contending for a plenist mechanism in which the human soul is a genuine substance but not a body.[3] Digby proposed rarefaction and condensation as the explanation of various physical phenomena in his *Treatise on the Nature of Body*, and it is interesting to consider his presentation of the doctrine in conjunction with Hobbes's conception of body. According to Digby, the nature of quan-

3. For more on Digby's natural philosophy, see Lasswitz 1890, 2:188–207, and Henry 1982.

tity "is nothing else but divisibility; and . . . a thing is bigge, by having a capacity to be divided, or (which is the same) to have partes made of it" (Digby 1644, 9). However, there are obvious cases of bodies that have the same volume but different weights, and the difficulty is to account for this fact. As Digby puts it, "Our measures tell us that their quantities are equall; and reason assureth us, there can not be two bodies in one and the same place; therefore when we see that a pinte of one thing outweigheth a pinte of an other that is thinner, we must conclude that there is more body compacted together in the heavy thing then in the light. . . . But how this comprehension of more body in equal roome is effected, doth not a little trouble Philosophers" (Digby 1644, 16). One proposed solution to this problem is to accept the concept of a vacuum, and explain density as the proportion of matter to empty space in a given body. But Digby accepts Aristotle's argument against the possibility of a vacuum in book 4 of the *Physics*, and regards it as "perfectly demonstrated, that no vacuity is possible in nature; neither great nor little: and consequently, the whole machine raysed upon that supposition, must be ruinous" (Digby 1644, 21).

Digby's solution to the difficulty requires a distinction between quantity and the substance that has quantity, and then defines dense bodies as those having relatively less quantity contained within their substances, while rare bodies contain more quantity in the same amount of substance:

> Thus then; remembering how we determined that Quantity is Divisibility: it followeth, that if besides Quantity there be a substance or thing which is divisible; that thing, if it be condistinguished from its Quantity or Divisibility, must of it selfe be indivisible: or (to speake more properly) it must be, not divisible. Putt then such substance to be capable of the Quantity of the whole world or universe: and consequently, you putt it of it selfe indifferent to all, and to any part of Quantity: for in it, by reason of the negation of Divisibility, there is no variety of partes, whereof one should be the subject of one part of Quantity, or another of another; or that one should be a capacity of more, another of lesse.
>
> This then being so, wee have the ground of more or less proportion between substance and quantity: for if the whole quantity of the universe be putt into it, the proportion of Quantity to the capacity of that substance, will bee greater then if but halfe that quantity were imbibed in the same substance. And be-

> cause proportion changeth on both sides by the single change of onely one side: it followeth that in the latter, the proportion of that substance to its Quantity, is greater; and that in the former, it is lesse; howbeit the substance itselfe be indivisible. (Digby 1644, 22)

If the universe had a perfectly homogeneous distribution of quantity throughout its whole extent, there would be no distinction between rare and dense bodies and the proportion of substance to quantity would be everywhere uniform. However, we observe that natural bodies (air, water, mercury, wood, gold, etc.) differ in their relative density and rarity, and this is accounted for by the fact that substance and quantity are not always in the same proportion throughout the natural world. In its application to the explanation of natural phenomena, this doctrine accounts for the greater divisibility of rare bodies, since they have more quantity (i.e., divisibility) in the same amount of substance as denser bodies. Similarly, the resistance of rare bodies is less than that of dense bodies, by virtue of their greater divisibility. A further virtue of this doctrine (at least in Digby's estimation) is its perfect concurrence with Aristotle's doctrine: "hee telleth us, that that body is rare whose quantity is more, and its substance lesse; that, contrariwise dense, where the substance is more and the quantity lesse" (Digby 1644, 23). With such a distinction between rare and dense bodies, rarefaction can then be defined as a process that compresses more quantity into a body and condensation as one that removes quantity from it.

Hobbes was familiar with Digby's philosophy but regarded it as utter nonsense. In particular, Digby's regard for Aristotle (as well, presumably, as his Catholicism) made him an adherent of the despised "Schools" that figure so prominently in the "Kingdome of Darknesse" denounced in part 4 of *Leviathan*. Digby is clearly one of the targets of Hobbes's denunciation of rarefaction and condensation in *Leviathan*,[4] and although the two had maintained a cordial correspondence

4. In summarizing and ridiculing the doctrine of rarefaction and condensation, Hobbes uses language that could have come straight out of Digby:

> If we would know why the same Body seems greater (without adding to it) one time, than another they say, when it seems lesse, it is *Condensed* when greater, *Rarefied*. What is that *Condensed*, and *Rarefied*? Condensed, is when there is in the very same Matter, lesse Quantity than before and Rarefied, when more. As if there could be Matter, that had not some determined Quantity; when Quantity is nothing else but the Determination of Matter; that is to say of Body, by which we say one Body is greater, or lesser than another, by this, or thus much. Or as

in the 1630s, Digby later expressed admiration for Ward's and Wallis's campaign against Hobbes.[5]

In any event, Wallis's use of the doctrine of rarefaction and condensation as a weapon against the Hobbesian philosophy of mathematics does not require that such processes be part of the natural order, but merely that they be possible or conceivable. By ruling out rarefaction and condensation as possible explanations of natural phenomena, Hobbes in effect makes his whole philosophy of mathematics depend upon the resolution of a natural-philosophical dispute. Wallis gleefully pointed out the extent to which Hobbes's account of mathematics was enmeshed in controversial physical principles, and he took this as further evidence that Hobbes had mistakenly conflated mathematics with physics:

> But, whether that opinion of Rarefaction and Condensation be true or not: yet since you cannot deny, but that it is at least a considerable controversy, and, by men as wise, and as good Philosophers as M. Hobs, maintained against you: yea and a Controversy not belonging to Mathematicks, but Physicks, or Naturall Philosophy, and there to be determined; it was not wisdome to hang the whole weight of Mathematicks upon so slender a thread, as the decision of that controversy in Naturall Philosophy, which whether way it be determined, is wholly impertinent to a Mathematical Definition. (*Due Correction* 56–57)

Even worse, because Hobbes also denies the possibility of a vacuum, he is apparently left without the resources to explain many natural phenomena (such as evaporation, the freezing of water, etc.) that others had accounted for either by rarefaction or the vacuum.[6] This is

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> if a Body were made without any Quantity at all, and that afterwards more, or lesse were put into it, according as it is intended the body should be more or lesse Dense. (*L* 4.46, 375; *EW* 3:678–79)

5. Four gracious letters from Digby to Hobbes date from 1636 to 1637; Digby's later hostility toward Hobbes is evident in a letter to Wallis from August of 1657, when he writes: "I must not take leave of you, till I have spoken a word or two of your worthy Colleague Doctor Ward. It is some time since I have heard of his booke against M. Hobbes. . . . It is a worthy Triumvirate that you two and Doctor Wilkins do exercise in literature and all that is worthy" (Wallis 1658, 11). Ward makes numerous favorable references to Digby in his *Exercitatio* (see *Exercitatio* 75, 93, 99, 161, and 200), and it is clear that, however idiosyncratic his theological or scientific doctrines, he was regarded favorably by the Oxford scientific establishment.

6. As Wallis observes, "[H]ow to salve these *Phaenomena*, (with many others of the like kind) without either *Vacuum*, which you deny, or *Condensation*, which you laugh

doubly problematic for Hobbes because, in addition to grounding his philosophy of mathematics in disputable (indeed, disputed) physical principles, he appears also to have forsworn the ready means of constructing a mechanistic account of the workings of nature.

Hobbes's defense against these charges was to insist that the disputes over rarefaction and condensation were not, properly speaking, physical debates, but rather verbal quarrels with no relevance to the foundations of mathematics. The empty terms *rarefaction* and *condensation* were used by "Schoole Divines" who did not understand even the first elements of natural philosophy or mathematics, and their speculations were to be left entirely out of any account of the true grounds of mathematics and physics. It is in this sense that Hobbes can claim that

> nature abhorres even empty words, such as are (in the meaning you assign them), *Rarefying* and *Condensing*. And you would be as well understood if you should say (coining words by your own power) that the same Body might take up sometimes a greater, sometimes a lesser place, by *Wallifaction* and *Wardensation*, as by *Rarefaction* and *Condensation*. (*SL* 2; *EW* 7:225)

Schaffer portrays this part of the controversy as a political struggle over the authority and autonomy of mathematicians (or, more particularly, professors of mathematics) to define the fundamental terms of their discipline. He reads Hobbes as holding "that geometers [are] deprived of the right to define their own terms themselves" (1988, 286–87). The Savilian professors' dispute with Hobbes was therefore "a contest over the propriety of the separate authority in civil society of teachers, mathematicians, or divines," in which "Ward and Wallis tried to resist Hobbes's denial of their right to ground geometrical terms" (1988, 288).

Central to this account is Schaffer's interpretation of Hobbes's declaration in the second of the *Six Lessons* that "[i]t is not the work of a Geometrician, as a Geometrician, to Define what is Equality, or Proportion, or any other word he useth, though it be the work of the same man, as a man" (*SL* 2; *EW* 7:222). Schaffer takes this as imposing restrictions upon geometers, which Wallis allegedly opposed by "insist[ing] that geometers properly and necessarily had the right to lay

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at; (one of which others use to assigne) because you find it too hard a task for you to undertake, (as well you may,) you leave to a *melius inquirendum*" (*Due Correction* 56).

down definitions and then use the privileged principles of geometric reason upon them" (Schaffer 1988, 287).

In reading the dispute this way Schaffer misinterprets the terms of the debate in order to reduce it to an entirely political struggle over the authority to define terms. Hobbes does *not* hold that the geometer (or anyone else) is forbidden to introduce definitions. Rather, his point is that in propounding definitions one is not doing geometry, but engaging in an activity common to all men: the stipulation of how words are to be understood. When Wallis refers to the principle that a body always retains the same quantity as part of a "mathematical definition," Hobbes objects to the use of the term:

> It seems by this, that all this while you think it is a piece of the Geometry of *Euclide*, no less to make the Definitions he useth, then to infer from them the Theorems he demonstrateth. Which is not true. For he that telleth you in what sense you are to take the Appellations of those things which he nameth in his discourse, teacheth you but his Language, that afterwards he may teach you his Art. But teaching of Language is not Mathematick, nor Logick, nor Physick, nor any other Science; and therefore to call a Definition (as you do) Mathematicall, or Physicall, is a mark of Ignorance (in a Professor) unexcusable. (*SL* 2; *EW* 7:225)

Nothing here forbids the geometer from introducing whatever definitions may please him, and this is perfectly in keeping with Hobbes's doctrine that definitions (rather than explicating the essence of a thing) are "truths constituted arbitrarily by the inventors of speech" (*DCo* 1.3.9; *EW* 1:37). Hobbes therefore objects to the concept of a specifically mathematical definition, not because the mathematician lacks the standing to introduce a definition, but because there is nothing essentially mathematical about the act of defining terms.

The degree of arbitrariness inherent in Hobbes's account of definitions does not, however, mean that all definitions are created equal. Some definitions are better than others, and Hobbes insists that the best definitions are "those which declare the cause or generation of that Subject, whereof the proper passions are to be demonstrated" (*SL* 2; *EW* 7:212). Terms such as *rarefaction* lack any proper or coherent definition because (at least according to Hobbes) they could be defined only by joining together words of contradictory signification. Just as he takes the term *immaterial substance* to be a contradiction in terms, Hobbes interprets *rarefied body* as equivalent to "quantity greater in

quantity than itself"—a manifest absurdity. According to Hobbes, empty terms of this sort are part of the obfuscatory apparatus of school divinity, and there is no question that he saw his account of quantity as inimical to the designs of Oxford's Savilian professors.[7] Nevertheless, Schaffer errs in depicting this part of the dispute as centering on the authority of geometers to introduce definitions.

The conceptual gulf dividing Hobbes's and Wallis's philosophies of mathematics should be clear from the foregoing discussion. Their differences in the philosophy of mathematics were not confined to questions of the relationship between mathematics and natural philosophy, but I think the other contested points can in large measure be traced back to a fundamental difference over the extent to which mathematics should depend upon principles of body and motion. We can see this more clearly by investigating the two thinkers' polemics over the proper account of geometric ratios, to which I now turn.

4.2 THE THEORY OF RATIOS AND ITS INTERPRETATION

The second fundamental issue raised and debated by Hobbes and Wallis was the thorny question of the nature of ratios. The differences on this point were so great that the interpretation of the theory of (geometrical) ratios occupied a central place in their exchange of polemics for more than twenty years. Of course, Hobbes and Wallis were hardly the only two thinkers to contest this issue. Wallis himself authored a vituperative attack on the *Dialogus de proportionibus* of Marcus Meibom (Wallis 1657c, 3).[8] In fact, the philosophico-mathematical literature of the period contains many disquisitions on the subject.[9] In chap-

7. In Στίγμαι Hobbes is quite explicit on this point: "But I beleeve you will by degrees become satisfied that they who say the same Numerical Body may be sometimes greater, sometimes lesse, speak absurdly, and that *Condensation* and *Rarefaction* here, and *definitive* and *circumspective*, and some other of your distinctions elsewhere are but snares, such as School-Divines have invented . . . to entangle shallow wits" (Στίγμαι 1; *EW* 7:384–85).

8. In the dedication to his refutation, Wallis actually notes that he was delayed in publishing by the necessity of answering Hobbes: "[I]n the mean time it happened that Hobbes needed to be punished a second time, now for the errors and bad language he foolishly spewed forth *[effutivit]* against me in an English piece prompted by my *Elenchus*, in which I refuted his geometry. This castigation of Hobbes kept both me and the printer occupied for some time" (*OM* 1:231). Meibom's theory and its connection to these debates is outlined in Sylla 1984, 30–35.

9. Barrow's *Lectiones Mathematicae* are probably among the most important of these. Barrow's principal task in these lectures was to defend the Euclidean theory of

ter 3 I outlined some of the contested issues when I examined Hobbes's account of ratios, and we are now in a position to examine his differences with Wallis. The points at issue in the debate are complex in themselves, and this complexity is augmented by numerous terminological confusions. Nevertheless, the most important aspects of the controversy can be made reasonably clear, and in the process we can gain some understanding of what was at stake, both mathematically and philosophically, in the dispute.

The place to begin unraveling this part of the controversy is with Hobbes's intention of founding the doctrine of ratios on the principles of body and motion. We have already seen that Wallis rejects such an attempt as an unwarranted intrusion of physical principles into mathematics. But Hobbes's program did not simply lead him to place mathematics on a "physicalistic" foundation, for it also encouraged him to propound a strong thesis on the homogeneity of certain magnitudes that had traditionally been deemed heterogeneous. In particular, Hobbes was persuaded by his analysis of motion and the ratios of motions that time is homogeneous to a line. At work here is a principle I mentioned in chapter 3, namely that quantities that do not, at first sight, appear to be captured in the definition of quantity in terms of the three dimensions of body can, nevertheless, be represented or "exposed" by lines. Velocities, times, weights, etc. are all traditionally taken as (one-dimensional) quantities, and Hobbes holds that they can be treated as the quantities of lines by which they are represented (*DCo* 2.12.4–7; *OL* 1:125–26). On the basis of this understanding of quantities and their representation, Hobbes concludes that lines and times are homogeneous quantities because, in the analysis of uniform motion, time is represented by a line and this line is compared to a distance traveled by a moving body:

> Seeing that it has been shown that in uniform motion the lengths covered *[percursum]* are as the parallelograms made by multiplication of the impetus by the time, it will also be, by permutation, as time to length so time to length; and, universally, all the properties and permutations of proportions enumerated in chapter thirteen hold here as well. (*DCo* 2.16.2; *OL* 1:186)

This doctrine permits the "permutation" of terms in a proportion to form two new ratios; thus, if the proportion $T_1{:}T_2 :: S_1{:}S_2$ links the

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ratios, which he does by summarizing and critiquing nearly every alternative account. Another contributor to these disputes was Jacobus Fontialis in his *De idea mirabilis matheseos de Ente* [1660?], in Fontialis 1740, 437–512.

ratio of times elapsed to spaces covered in uniform motion, it can be permuted to form the new proportion $T_1:S_1 :: T_2:S_2$, in which the times and distances are compared directly with each other.

Wallis attacked this portion of Hobbesian doctrine as an intolerable confusion. Where Hobbes had taken the results from the mathematical analysis of motion to show that time and longitude are homogeneous, Wallis observed that the permutation of proportions is admissible only if all the quantities in the proportion are pairwise homogeneous. Thus, because time and length are not capable of direct comparison with one another, they are heterogeneous magnitudes and the terms in the proportion cannot be permuted.[10] Because of his commitment to the numerical conception of ratios Wallis was happy to allow that *ratios* are themselves quantities and, indeed, quantities homogeneous to one another; but he denies that the terms in any two given ratios need be homogeneous. He insists upon precisely this point in chapter 25 of the *Mathesis Universalis,* with the declaration

> All ratios of whatever quantities to one another are homogeneous among themselves. As for example lines and weights are clearly things heterogeneous, nor do they have a ratio to each other; but the ratio of a line to a line and of a weight to a weight are plainly homogeneous. For example, the ratio of a two-foot line to a one-foot line and that of a two-pound weight to a one-pound weight are homogeneous, and even the same ratio, namely double: for if the division is effected the quotient 2 will arise in both cases. (*MU* 25; *OM* 1:36)

It is with this conception of ratios and homogeneity in the background that Wallis can demand of Hobbes: "Tell us then, if you can, what ratio does one hour of time have to two ells of length? There is indeed some ratio of the adjoined numbers, one and two; but there is none of time to length, namely of an hour to an ell [an old English unit of measure, equal to approximately forty-five inches]" (*Elenchus* 41–42). For his part, Hobbes was intent upon maintaining a distinction between different species of magnitudes, and he did not accept that any two magnitudes were necessarily homogeneous. Nevertheless, because he saw the quantity of time as expressible by a line, he was prepared to opt for a general thesis of the homogeneity of time and line. In his

10. As Wallis puts it in the *Elenchus,* "And so there can be a ratio of time to time, and also of length to length (which ratios can either be the same or different), because times are homogeneous to times and lengths proportional to lengths, but there can be no ratio of length to time because these are heterogeneous" (*Elenchus* 41).

Six Lessons he grants that "Time and Line are of divers natures" (*SL* 4; *EW* 7:273) but insists that their quantities can be compared by straight lines, and in this sense they may be taken as homogeneous. To Wallis's demand that he declare what ratio an hour bears to an ell, Hobbes replies that it "is the same Proportion that *two Hours* have to *two Ells*," and by this retort he imagines to have shown Wallis that "your Question was not so subtile as you thought it" (*SL* 4; *EW* 7:273).

The driving force behind this unusual doctrine is Hobbes's insistence that all magnitudes are ultimately derivable from the three dimensions of body. In Hobbes's scheme, there must necessarily be heterogeneity of one-, two-, or three-dimensional magnitudes because lines, surfaces, and solids (the three fundamental dimensions of body) cannot be compared with one another as to quantity. On the other hand, Hobbes reduces such one-dimensional magnitudes as time, mass, density, and temperature to the case of lines, since it is by lines that these magnitudes are exposed and measured. Under criticism from Wallis, Hobbes rephrased this into a declaration of the homogeneity of the *quantities* of times and lines, rather than the homogeneity of times and lines themselves; but he retained the leading idea that all quantities must be traced back to the three dimensions of body. In the *Examinatio,* after declaring it "absurd to say that a stone is quantity, and no less absurd to say that time is quantity" (*Examinatio* 3; *OL* 4:118), Hobbes nevertheless insists that the weight of a stone or the length of time can be measured, and thus have quantity attributed to them. The result is that

> "How much" can be said of natural bodies as well as time, although neither of them can be called quantity abstractly. For all quantity, strictly speaking, is either longitude, or surface, or solid, or as one is accustomed to say, mathematical body. But time, motion, and force, and other things of which it can be asked how much they are, all have quantities, by which it is determined how great they are. And these things surely have one or another of those three dimensions, for the things themselves are measured by them. (*Examinatio* 3; *OL* 4:119)

The result of this doctrine is that "although it is absurd to say that a line is equal to a time, nevertheless it is not absurd to say that a quantity of time is equal to the quantity of a line" (*Examinatio* 3; *OL* 4:119).

Wallis was not alone in objecting to this feature of Hobbes's concep-

tion of ratios. Barrow poured scorn on the doctrine, complaining that it was grounded in a confusion of two fundamentally different senses of the word *measure*. As Barrow points out, a quantity can be said to be the measure of another (homogeneous) quantity when it is a part of the measured quantity, as when distances are measured in meters or time in minutes. It is in precisely this sense that Euclid speaks of measure when he declares that "a magnitude is a part of a magnitude, the less of the greater, when it measures the greater" (*Elements* 5, def. 1). Hobbes adopts essentially this usage in the *Examinatio* when he defines measure as "one magnitude of another, when it or multiples of it, applied to the other, coincide with it" (*Examinatio* 1; *OL* 4:18). On the other hand, a measure can be, in Barrow's words, "anything that may either conveniently represent or in some way signify another" (*LM* 16, 263), as when time is represented by a line. As Barrow observes, time can be measured in the first sense by minutes or hours (which are equal parts of time itself) or in the second sense by numbers, lengths, or any other method of keeping track of the passage of time. He concludes that if Hobbes "had seen clearly or weighed this ambiguity of the word *measure*, or if he had remained more constant to his own definition, he would not, I think, have so conflated the quantities of all things and been moved by such an argument to pronounce them homogeneous to one another" (*LM* 16, 264).

Barrow is very much in the right here, and his analysis of the problem shows some of the difficulties encountered in Hobbes's insistence that all quantities must ultimately be derived from the three fundamental dimensions of body. It is manifestly absurd for Hobbes to maintain the general thesis of the homogeneity of all one-dimensional magnitudes, since the "permutation" of proportions that he desires can be carried out only after the specification of units to link the otherwise heterogeneous magnitudes. For example, there is no way to connect length and temperature without first specifying a unit (degree) for the measure of temperature, another unit (millimeters, say) for the measure of distance, and a device (such as a mercury thermometer) that is so calibrated that one degree is represented by a specific distance (suppose one degree is represented by one millimeter on the temperature scale). Once such a specification of units has been achieved, it is a trivial matter to establish a correspondence between temperature and length, but the correspondence remains entirely dependent upon the choice of units. In order for the different magnitudes to be truly homogeneous, however, they would have to be capable of direct comparison with one another, and it is on this point that Hobbes's argument re-

duces either to absurdity or triviality. When he insists that "it is absurd to say that a line is equal to a time, nevertheless it is not absurd to say that a quantity of time is equal to the quantity of a line" (*Examinatio* 3; *OL* 4:119), Hobbes essentially grants his opponents' case. Because time and line cannot be compared directly, they are not homogeneous; but from the fact that their respective quantities can be compared, it follows only that when units have been assigned to one-dimensional magnitudes, these units are capable of comparison with one another.

The general issue of homogeneity of magnitudes, and the relationship between questions of homogeneity and proper mathematical procedures, will occupy us again when we consider the problem of the angle of contact and Hobbes's rejection of analytic methods. For present purposes, however, we can leave the issue aside and return to a more specific consideration of the controversy over the general doctrine of ratios and proportionality.

Hobbes's attempt to define ratios in terms of bodies was intended, in part, to resolve serious questions about the proper interpretation of the theory of ratios. His program of taking ratios as expounded by bodies treats ratios as essentially geometrical, and he resists the "numerical" understanding of ratios in which the ratio of two magnitudes is treated as a quantity and, at least on some presentations of the theory, identified with the quotient of the magnitudes that form the ratio. For his part, Wallis was thoroughly committed to the numerical conception of ratios, and this naturally brought him into conflict with Hobbes's doctrines.

According to Wallis, the numerical theory of ratios is both more easily understood than the relational theory and more general in its application. In particular, he argued that his treatment of ratios as a generalized kind of quotient allows for otherwise complex comparisons of ratios to be reduced to the analysis of quotients. "The quotient of division," he declares in the *Mathesis Universalis*,

> shows the ratio of the dividend to the divisor. So if 12 is divided by 6, it will produce the quotient 2, the ratio commonly known as *double*, which is the ratio that the number 12 has to the number 6. And also a four pound weight is double a two pound weight, because if four pounds are divided by two pounds, the quotient 2 will be produced, or how often *[quoties]* the one weight is contained in the other. And because of this fact, *Where quotients are equal to one another, then also the quantities forming the quotients are in the same ratio.* And indeed the ratio is

> estimated by the quotient, and also from the equality of ratios there follows the equality of quotients. (*MU* 25; *OM* 1:135)

As part of his extended case for the priority of arithmetic over geometry, Wallis devoted chapter 35 of the *Mathesis Universalis* to an "arithmetical" presentation of the theory of ratios from book 5 of the Euclidean *Elements.* The theory of ratios and proportions, he explains, "is arithmetical rather than geometrical," providing that arithmetic is understood to encompass more than simply the theory of integers and includes "fractions, surds, the whole of algebra, and (as it is called) the specious or symbolic arithmetic" (*MU* 35; *OM* 1:183). Given this expansive conception of arithmetic, it is no surprise to find Wallis identifying numbers and ratios with the declaration that:

> the whole of arithmetic, if it is regarded more strictly, appears to be scarcely anything other than the doctrine of ratios. And numbers themselves are so many signs of ratios whose common consequent is one, or unity. And here one or unity is taken for an exposed quantity, so that all other numbers (whether integers, fractions, or even surds) are so many signs or exponents of other ratios to an exposed quantity. (*MU* 35; *OM* 1:183)

According to Wallis, the ancients preferred a cumbersome and particularized form of expression when they represented ratios by pairs of lines rather than numerical signs, and in effect they confined the theory of proportions to geometry. This shortcoming was due to an insufficiently general conception of number, which arose principally "because the symbolic method did not then obtain, and even the numerical notation in use everywhere today (and invented by the Indians) had not yet been introduced" (*MU* 35; *OM* 1:183). However, once the resources of symbolic algebra have been brought to bear, a completely general theory of ratios emerges, in which "the quantities of lines, or plane figures, or even solids, or anything else" are represented "by the letters of the alphabet" (*MU* 35; *OM* 1:183).

The scheme proposed by Wallis involves taking the ratio of two magnitudes as a quotient or a number representing how many times the antecedent is contained in the consequent. At first sight, this treatment of ratios as quotients would seem to restrict the theory of proportion to the realm of rational numbers, but Wallis maintained that the generalized understanding of number implicit in the new algebra would expand the concept of quotient to encompass both rational and irrational magnitudes.

This identification of ratios with quotients was not a complete innovation on Wallis's part. As I noted in chapter 3, the numerical theory of ratios had a long history before Wallis. Moreover, the treatment of ratios as quotients is implicit in the numerical theory, since it takes ratios to have "sizes" or "denominations" that are most easily understood as quotients arising from the division of two magnitudes. Before Wallis, the authority of Oughtred's *Clavis Mathematicae,* for example, could be used to underwrite the conception of ratios as quotients, because Oughtred is willing to identify ratios with the result of division. According to his chapter "Of Proportion":

> If of four given numbers the first is to the second as the third to the fourth, these four numbers are said to be proportional. And the relation *[habitudo]* of numbers one to another is found by dividing the antecedent by the consequent: as 31 to 7 is the ratio 4 3/7, that is quadruple supertripartient seven. (Oughtred 1693, 15)

This general account of proportion speaks of "numbers" standing in the same ratio when, by division, they are found to have the same relation, which is to say that they form the same quotient. For Oughtred, numbers are not simply positive integers, but any kind of magnitude at all, and he sees no deep difference between numbers and algebraic "species" or letters that can indifferently denote any kind of quantity. As he notes at the beginning of the *Clavis Mathematicae,*

> Magnitudes can be denoted either by numbers signifying their measure, or also by species: as a line seven inches long is designated by 7; or by any one letter or note as *A, B, C,* etc; or by two letters assigned to the termini of the line, as *AB, BC, CD,* etc. And any of these can be taken at will. But you must retain in memory the magnitude for which any species is intended. (Oughtred 1693, 3–4)

By assimilating the theory of ratios into a generalized treatment of number Oughtred, Wallis, and other proponents of the numerical conception of ratios effectively abandoned the classical conception of ratios as a special kind of relation between magnitudes. In their treatment, ratios become quantities ultimately reducible to number.

As we saw in chapter 3, Hobbes rejected this understanding of ratios, and accused Wallis of propagating a serious confusion by identifying ratios with quotients. In the dedicatory epistle to the *Six Lessons,* Hobbes lists among the principles "so void of sense, that a man at the first hearing, whether Geometrician or not Geometrician, must

abhor them," the principle that "the Quotient is the Proportion of the Divisor [reading "Division" in Hobbes's original as "Divisor"] to the Dividend" (*EW* 7:186–87). On Hobbes's analysis, a quotient can only be found "in *aliquot parts*," and the general theory of proportions must be phrased in terms that can accommodate incommensurable ratios. He insists that "setting their Symboles one above another doth not make a Quotient," and argues that the presence of incommensurable magnitudes make it "impossible to define Proportion universally, by comparing Quotients" (*SL* 3; *EW* 7:241). Hobbes further objected to the identification of ratios with quotients on the grounds that a quotient or fraction such as ⅔ is an absolute or determinate quantity that cannot be conflated with a ratio, because a ratio is always a relative measure of two quantities that expresses how great one is with respect to another.[11]

Wallis replied to such objections by disingenuously insisting that he had always been scrupulous in observing a distinction between ratios and quotients. One such response to Hobbes's accusations is worth considering at length, for it also gives some idea of the rhetorical standards observed by the two disputants:

> But you adde farther, that *I say, that I make [ratio] to consist in the Quotient*. And is not this abominably false? I neither say so, nor doe so, nor did I give any ground at all for any man (that is in his witts) to believe I did. My words were these, *Videmus igitur Rationis aestimationem esse (secundum Te) penes Residuum, non penes Quotum, & Subductione, non Divisione quaerendam esse*. (And what reason I had to say so, they that consult the place will see.) Now could any man (who had not a great confidence that his English Reader understands no Latine) be so impudent as to say, that in those words, *I say, you make Proportion to consist in the Remainder; and I, in the Quotient?* Can any man, that understands, though but a little Latine, (if he be not either out of his witts, or halfe asleep,) think that these words *Rationis aestimatio est penes Quotum,* (that is, *the Proportion is to be estimated according to the Quotient*, or, to use your own words, *the quotient gives us the measure of the proportion*,) could thus

11. This is the import of Hobbes's declaration at the opening of chapter 13 of *De Corpore*, where he defines ratios after first observing that "great and little are intelligible only by comparison. Now that to which they are compared we have called something exposed; that is, some magnitude either perceived by sense, or so defined by words that it can be comprehended by the mind" (*DCo* 2.13.1; *OL* 1:128).

> be Englished *proportion consists in the quotient*? And that then you should raile at us, quite through your Book for saying that *Proportion* is *a certain quotient*, that it is a *number*, that it is an *absolute quantity*, &c. as if we had been so ridiculous as to speak like you. For, that you have so spoken you cannot deny, (and therefore the absurdity what ever it be, lights upon your selfe:) But, to say that *I said so*, or any thing to that purpose, till you can shew where I said it, I take to be, (so farre as a word out of your mouth can be) a *manifest slander*. I neither say so, nor think so. (*Due Correction* 62–63)

This lengthy and vociferous reply repeats an earlier allegation from Wallis's *Elenchus* to the effect that Hobbes absurdly treats ratios as numbers formed by the subtraction of one quantity from another. As I showed in section 2 of chapter 3, this objection depends upon an uncharitable reading of Hobbes's definition of ratio and a failure to take seriously his distinction between arithmetical and geometrical ratios.[12] More interesting is Wallis's heated denial of ever having identified ratios and quotients, because it seems to contradict his other pronouncements on ratios. It may perhaps be true that, strictly speaking, Wallis's works do not contain the explicit formulation "The ratio of two magnitudes is the quotient arising from their division" (or its Latin equivalent). But from what we have seen of his remarks on ratios, it is clear that he does, indeed, take ratios to be quantities falling within the purview of a generalized science of arithmetic. Where earlier authors had confined quotients to the division of integers, Wallis desires to expand the concept of arithmetic to include quotients of irrational

12. Where Hobbes had declared in *De Corpore* that "the relation of the antecedent to the consequent according to magnitude, that is to say its equality, excess, or defect is called the ratio and proportion of the antecedent to the consequent; and so ratio is nothing other than the equality or inequality of the antecedent compared to the consequent according to magnitude" (*DCo* 2.11.3; *OL* 1:119), Wallis commented, "But since the quantity, for example, of inequality can be considered either with respect to the remainder, or with respect to how often *[quoad Quotum]* (that is to say, it can be inquired either how much greater this is than that, or how often or how much this is of that; the first of which is made known by subtraction, the second by division) let us see which of these ways of consideration is that which you want to call ratio" (*Elenchus* 14). He then takes Hobbes's remark that "ratio consists in the difference of the antecedent to the consequent, that is in that part of the greater remaining after having taken away the lesser" (*DCo* 2.11.5; *OL* 1:119) to show that Hobbes identifies ratios universally with differences. But, of course, Hobbes distinguishes between arithmetical ratios and geometric ratios in terms not unlike those used by Wallis, and although Hobbes can be sloppy in his use of the term "difference" it is clear that he is not committed to the view to which Wallis objects.

magnitudes, and it is this "larger sense" of quotient that he uses when treating ratios as quotients.[13]

Wallis's insistence that he never actually takes ratios to be quotients is, in effect, a verbal dodge. Not willing to appear to depart too radically from the authority of Euclid and classical geometry, Wallis grants that ratios are "exposed," "determined," "denominated," "designed," or "estimated" by quotients while denying that ratios and quotients are the same thing. But it is difficult to attach much sense to such a distinction between ratios and quotients. All of the relevant facts about ratios are facts about the quotients that "expose" the ratio, and (apart from preserving the appearance of fidelity to the Euclidean tradition) there is no point in insisting upon the distinction. Hobbes took note of this very point, observing that Wallis's pronouncements on his theory of ratios did not sit well with his words in *Arithmetica Infinitorum.*[14] In Wallis's presentation, the ancient theory of ratios has essentially been subsumed into an algebraic theory of numbers, and the numbers corresponding to ratios are in fact quotients.[15] As their dis-

13. Wallis makes this quite explicit in chapter 9 of his *Treatise of Algebra,* where, in discussing the Euclidean definition of ratios, Wallis announces that the "whole Definition of λόγος (Ratio, Rate, or Proportion) . . . is thus rather to be rendered, . . . *Rate* (or Proportion) *is that Relation of two Homogeneous Magnitudes* (or Magnitudes of the same kind,) *how the one stands related to the other, as to the* (Quotient, or) *Quantuplicity*" (*Treatise of Algebra,* 1685, 79). Later, he acknowledges that this requires an expanded conception of the quotient:

> But all other Proportions which they call Ineffable, (which are not *ut numerus ad numerum,*) but as Quantities Incommensurable, and for the sake of which, that Scholiast tells us, that Euclide chose to use the word πηλικότητες rather than ποσότης, (for what we commonly call the Quotient in the largest sense) that it might extend to Ineffable as well as Effable Proportion, (as if in Latine he would have said Quantuplum, rather than Quotuplum, lest this should be thought to extend only to Multiples, or but to Effable Proportions;) all these, I say have no peculiar Names allotted; but use to be designed by the Terms themselves, as A to B, or as 1 to √2, or (set Fraction-wise, so as to design a Quotient,) A/B, or 1/√2 &c. (*Treatise of Algebra* 9, 80)

14. In commenting upon the manipulation of ratios in *Arithmetica Infinitorum,* Hobbes remarks, "But first I wonder why you were so angry with me for saying you made proportion to consist in the Quotient, as to tell me it was abominably false, and to justify it, cite your own words *Penes Quotientem;* do not you say here, the proportion is everywhere greater than subtriple, or ⅓? And is not ⅓ the quotient of 1 divided by 3? You cannot say in this place that *Penes* is understood; for if it were expressed you would not be able to proceed" (Στίγμαι 1; *EW* 7:366).

15. Klein, in commenting upon this aspect of Wallis's general theory of number remarks that "the universality of arithmetic as a 'general theory of ratios,' which depends on the homogeneity of all 'numbers,' can be understood only in terms of a symbolic reinterpretation of the ancient 'numbered assemblage,' of the *arithmos.* The object

pute dragged on over the years, Hobbes and Wallis continued to trade accusation and denial on this issue with no significant change in position,[16] and we can leave this part of the quarrel over ratios aside and turn to a consideration of related issues concerning the theory of proportion.

The dispute over the nature of ratios naturally carried over into the thorny issue of the compounding of ratios—a troublesome point that had been the source of debate and confusion well before Hobbes and Wallis. The fundamental difficulty can be illustrated by first considering a pseudo-Euclidean definition that made its way into some editions of book 6 of the *Elements*. The fifth definition of book 6 reads:

> A ratio is said to be compounded of ratios when the sizes (πηλικότητες) of the ratios multiplied together make some (?ratio, or size). (*Elements* 6, def. 5)[17]

The remarkable features of this definition are its reference to the "sizes" of the compounded ratios and its application of the arithmetical operation of multiplication to construct a ratio from the sizes of two given ratios. According to the relational theory, a ratio is not a quantity and thus lacks a size, so it is difficult to reconcile this definition with the "official" presentation of the theory of ratios in book 5 of the *Elements*. As Barrow points out, "ratios, as they lack all quantity, can neither be added nor multiplied" (*LM* 20, 326). Moreover, the

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of arithmetic and logistic in their algebraic expansion is now defined as 'number,' and this means a symbolically conceived ratio—a conception consonant with that of algebra as a general theory of proportions and ratios" (1968, 223).

16. Thus, for example, in the *Examinatio*, one of the interlocutors in the third dialogue reports that "often, as he does here, Wallis says that fractions or quotients are the same thing as ratios. But when he was warned by Hobbes that fractions and quotients are all absolute quantities, but ratios all comparative quantities, he denied that he had said that a quotient is the ratio itself, but that the ratio is according to the quotient. But he attempts to base his whole treatise *Arithmetica Infinitorum* on this foundation: that the quotient itself is the ratio of the dividend to the divisor, so that ⅓ is the ratio of 1 to 3" (*Examinatio* 3; *OL* 4:128). Wallis's rejoinder in *Hobbius Heauton-timorumenos* was to claim, "No, I doe not make *Proportion, a Quotient,* or *an absolute Quantity* (that's but his inference, and a weak one.) I say indeed that Proportion *depends upon* the Quotient, is *determined* by the Quotient, *estimated* by the Quotient, and *denominated* by the Quotient; not that it is the Quotient" (1662, 51).

17. The reasons for regarding this definition as a late interpolation are mentioned briefly below. For a more complete discussion, see Heath's commentary in Euclid [1925] 1956, 2:189–90. Sylla 1984, 22–24, also discusses some of the problems with the definition. Even though the definition is not, strictly speaking, part of the Euclidean corpus, it will be more convenient if I refer to it as definition 5 of book 6 of the *Elements*, since that is how it was known to Hobbes, Wallis, and their contemporaries.

definition is never used in the *Elements,* even in the one place where Euclid speaks of compound ratios (*Elements* 6, prop. 23).[18] Henry Savile characterized the definition as one "of the two moles or blemishes on the most beautiful body of geometry" that had engaged the attention of ancient and modern geometers (Savile 1621, 140). The other famous blemish was the parallel postulate, and Savile thought that both should be remedied by showing that they are demonstrable from the remaining Euclidean principles.

For all its theoretical problems in regard to the relational theory of ratios, however, definition 5 makes perfect sense in conjunction with the numerical theory of ratios. If ratios are endowed with sizes, or if they are simply identified with quotients, then it is a simple matter to accept the idea that multiplication can generate a new ratio from two given ratios. Thus, if the ratios 3:7 and 1:4 are compounded, the new ratio that emerges is that found by multiplying the quotients $3/7$ and $1/4$, yielding $3/28$ or 3:28 as the compounded ratio.

Wallis was eager to solve the problems associated with the fifth definition of book 6, and attempted to show that any confusion or obscurity surrounding it could be overcome by, in effect, showing that Euclid himself accepted the numerical theory of ratios. This strategy is clear in his essay "The Geometrical Dispute over the Fifth Postulate and Fifth Definition of Book 6 of Euclid," where Wallis undertakes to rectify the two flaws that Savile had complained of as disfiguring the body of Euclidean geometry. The essay apparently originated as a pair of Savilian lectures in 1651 and 1663, although it was not published until 1693 in volume 2 of Wallis's *Opera Mathematica.*[19] Wallis's treatment of definition 5 proceeds by first claiming that Euclid's definition of ratios as "a sort of relation in respect of size between two magnitudes of the same kind" (*Elements* 5, def. 3) should be understood somewhat differently than the tradition takes it. In particular, he coins the neologism *quantuplicity* and takes the Euclidean definition of ratios to be the following: "A ratio is that relation or habitude of homo-

18. This asserts, "Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides." Euclid does not actually define the term *compound ratio,* but from the context of the proof in *Elements* 6, prop. 23, the meaning is clear enough: If *K:L* and *L:M* are ratios, then the ratio *K:M* is the ratio compounded of the ratio of *K* to *L* and that of *L* to *M.* In any event, it is noteworthy that the so-called fifth definition from book 6 does not appear in the proof.

19. Prag 1931, 402, reports that "[a]s Savilian Professor at Oxford Wallis was required to work on Euclid's axiomatics, above all on the parallel postulate. In two lectures, February 1651 and July 1663, he fulfilled this obligation." Prag refers here to the essay in Wallis's *Opera* (2:665–78).

geneous magnitudes to one another in which it is shown how the one is to the other, considered according to quantuplicity" (*OM* 2:665). The slippery term *quantuplicity* is Wallis's word for how much one magnitude is in comparison with another, or how many times the one is contained in the other. In particular, he wants to allow relations of quantuplicity that cannot be expressed by ratios of integers, so that the side of a square would be the $1/\sqrt{2}$ quantuple part of the diagonal.[20]

With this (strongly numerical) understanding of ratios in place, Wallis proceeds to interpret the fifth definition of book 6, asserting that the compounding of ratios arises from the multiplication of the quotients that "expose" the compounded ratios. The definition, he says, "is to be understood this way: *A ratio is said to be compounded of ratios when the exponents of the ratios multiplied together make the exponent of that ratio*" (*OM* 2:666). This requires that Euclid's term πηλικότητες (*Elements* 6, def. 5) be taken to designate the quantities or exponents of the compounded ratios, which quantities are therefore nothing other than the quotients of the antecedents and consequents in the ratios. Understood in this way, the definition is no "flaw" to be removed from geometry (as Savile had thought), but simply a stipulation of how the expression *compound ratio* is to be used.[21]

Hobbes naturally resisted such an account of compounded ratios because he saw it as depending too closely on the numerical theory of ratios. In his understanding of the fifth definition in book 6 of the *Elements*, Hobbes holds that Euclid intended the term πηλικότητες to apply, not to the quantities of the compounded ratios, but rather to the quantities that form these ratios. In other words, where Wallis takes

20. The linguistic difficulties surrounding Wallis's pronouncements on this point are sufficiently grave to render all but hopeless any attempt at translation into English. He distinguishes quotuplicity from quantuplicity in much the same way that other authors distinguish aliquot from proportional parts of a magnitude: quotuplicity is a relation of one magnitude to another that can be expressed by a fraction (or, equivalently, a ratio of integers), but quantuplicity is the more general relation between magnitudes that need not be expressible in rational terms. Thus, he contends, "Hoc est, *Quotupla* sit, seu potius *Quantupla*, altera alterius. Quod sic intellectum volo: Nimirum, prout *Pars Aliquanta* distingui solet a *parte Aliquota;* sic ego (quae sunt earum Correlata) *Aliquantuplum* ab *Aliquotuplo* distinguo. Ut *Quantuplum* sit vox Generalis, cujus *Quotuplum* sit un Species, qui *Multiplum* respondeat. . . . Quae Relatio apud Graecos innui solet terminatione πόσιον apud Latinos terminatione *-plum*, apud Nos terminatione *-fold*" (*OM* 2:665).

21. As Wallis puts it, "Therefore there is no reason why this is a demonstrable proposition (any more than are all other definitions), for nothing more is taught here than in what sense he desires this locution to be understood. . . . And therefore there is in this definition no mole to be removed, or blemish to be cleansed away" (*OM* 2:666).

the quantities of the ratios *A:B* and *C:D* to be the quotients *A/B* and *C/D*, Hobbes understands the magnitudes *A, B, C, D*, taken singly, to be the quantities of the ratios.[22] This may seem to be a relatively minor difference of opinion over how best to render a Greek term into mathematical English, but it amounts to nothing less than the contest between the numerical and relational theory of ratios.

This disagreement also set the stage for terminological confusions that would persist throughout the dispute. Following the older tradition and its usages, Hobbes regarded the compounding of ratios as an operation of addition rather than multiplication, since the compounding to two ratios is effected by, so to speak, taking them together. Earlier authors had spoken of the compounding as "addition,"[23] and Hobbes's conception of the matter is not far removed from this earlier view. As it happens, the compound ratio can be found by multiplying the antecedents and consequents of the two compounded ratios, and this fact led to a great deal of the linguistic and conceptual confusion over whether the process of compounding should be characterized as multiplication or addition.[24]

22. As Hobbes puts it in *De Corpore*, "Of two ratios, whether arithmetical or geometrical, when the magnitudes compared in both (which are called by Euclid, in the fifth definition of his sixth book, the quantities of ratios) are equal, then one ratio can be neither greater nor less than the other; for one equality is not greater or less than another equality" (*DCo* 2.13.3; *OL* 1:129–30).

23. See Sylla 1984, 11–20, on this usage, which persisted until roughly the time of Newton.

24. Barrow reports that "because when ratios are said to be compounded their denominators are multiplied, it is manifest that ratios are more rightly said to be multiplied than added. Just as when one of the denominators divides another, such an operation will be more justly called the division than the subtraction of a ratio. Nevertheless it has obtained that the former operation is called addition [*πρόσθεσις*], and the latter subtraction [*ἀφαίρεσις*]" (*LM* 20, 326–27). In a letter to John Collins from 8/18 September 1668, Wallis commented on the linguistic confusion as follows:

> That which I could wish altered in Mr. Mercator's logarithmotechnica was his manner of expression of the composition of ratios. For composition be a word used by Euclid sometimes for addition, sometimes for multiplication, and there being in him two compositions of ratios, the one mentioned in Def. 14, Lib. 5, which is by addition of the exponents, as when 3/2 + 2/2 = 5/2, the other by multiplication of the exponents, Def. 5, Lib. 6, as where 3/2 × 2/2 = 6/4, which ambiguity hath caused some confusion, especially where the latter is called an addition of ratios. Clavius, and Gregory St. Vincent, and divers others, to avoid this inconvenience, have, for distinction sake, called the former composition by addition, the latter composition by multiplication; with which most writers, who speak distinctly, have used to comply: of which I have spoken at large in what I have writ against Meibomius, and against Mr. Hobbes's fourth dialogue, and elsewhere" (Wallis to Collins in Rigaud [1841] 1965, 2:494–95).

Wallis ridiculed Hobbes's account of the compounding of ratios for its alleged terminological inadequacies, suggesting that Hobbes was ignorant of the most basic operations of arithmetic and their application to the theory of ratios. In Hobbes's presentation, the equivalent of the fifth definition of book 6 of Euclid can be proved as a theorem rather than introduced as a definition. In his parlance, it reads:

> If there are any three magnitudes, or any three things that have some ratio one to another, as three numbers, three times, three degrees, etc., the ratios of the first to the second and of the second to the third, taken together, are equal to the ratio of the first to the third. (*DCo* 2.13.13; *OL* 1:140)

Wallis objected that, instead of speaking of the ratios "taken together," Hobbes should have called the result a "compounded ratio," and in any case the presentation confuses addition with multiplication:

> Let there, for example, be these three quantities: the numbers 6, 3, 1. The ratio of the first to the second is double, of the second to the third triple, and the ratio of the first to the third sextuple: but it is not as you intend, that double and triple taken together equal sextuple (yet these words seem to be understood, when you say that the first and second ratios taken together are equal to the third). For this is completely false, as double and triple taken together are equal (not to sextuple, but) to quintuple. (Although perhaps it could be doubted whether or not you sufficiently understand this.) But by the first and second ratios taken together, you understand the ratio compounded of the first and second, which composition is nevertheless done by the multiplication of the terms (not their addition), as is well known. And this should be called the continuation of ratios rather than their addition (although I do not deny that this appellation can be found in some places). And so the ratio that is compounded, for example, of double and triple is not the double and triple taken together (which is quintuple, because $2 + 3 = 5$) but the triple of the double, that is sextuple, because $2 \times 3 = 6$, in which sense your proposition is true and nearly identical with the fifth definition in the sixth book of Euclid. (*Elenchus* 20)

Hobbes's response to this criticism was to insist that Wallis did not understand the distinction between ratios and numbers. This accusation is, of course, related to his earlier charge that Wallis fails to distin-

guish between ratios and quotients. Replying to the specific example used by Wallis, Hobbes asks:

> Tell me (egregious professors) how is six to three double Proportion? Is six to three the double of a number, or the double of some Proportion? All men know the number six is double to the number three, and the number three triple to an unity. But is the Question here of compounding numbers, or of compounding proportions? . . . Your instance therefore of six, three, one, is here impertinent, there being in them no doubling, no tripling, no sextupling of Proportions, but of numbers. (*SL* 3; *EW* 7:244–5)

This complaint could obviously have little persuasive effect on Wallis, whose entire approach to the theory of ratios was predicated upon the thesis that there is no deep distinction between numbers and ratios. Wallis takes the number 3 to represent the ratio 3:1, and he simply denies Hobbes's assumption that there is a difference between doubling numbers and doubling proportions.

Aside from its lack of convincing force to a staunch adherent of the numerical theory of ratios, Hobbes's account of ratios had the disadvantage of introducing yet another layer of verbal confusion into the debate. This time, the issue revolves around the question of the difference between such terms as *double* and *duplicate,* and their application to ratios.[25] Classically, in a continued proportion such as $X:Y :: Y:Z$, the magnitudes X and Z are said to be in duplicate ratio, with the idea being that the original ratio of X to Y appears twice, or is duplicated, in the proportion. The ratio $X:Z$ (which, by the way, is compounded of the ratio $X:Y$ and $Y:Z$) can then be said to be the duplicate ratio of the original ratio $X:Y$. On the other hand, a magnitude is double of another when it is twice as great. In exactly the same manner, triplicate ratio can be distinguished from triple quantity, quadruplicate from quadruple, and so forth. The subduplicate of a ratio is obtained by interposing a quantity in continued proportion between the antecedent and consequent of the original ratio; i.e., the subduplicate of the ratio $A:B$ is found by interposing a quantity C such that $A:C :: C:B$. Thus, the ratio $1:\sqrt{3}$ is subduplicate of the ratio 1:3, since $1:\sqrt{3} :: \sqrt{3}:3$.

Hobbes did not follow the established usage, nor did he remain consistent in his own terminology. In the Latin version of *De Corpore* he declared that "the ratio of a greater to a lesser quantity is multiplied by a number when a certain number of ratios equal to it or the same

25. See Saito 1993 for an account of duplicate ratios in Euclid.

are added together"; but when the ratio of a lesser to a greater quantity is iterated, "it is not properly said to be multiplied, but submultiplied" (*DCo* 2.13.16; *OL* 1:145). In the English version of *De Corpore* he reverted to the more traditional usage in which "[a] Proportion is said to be multiplied by a Number when it is so often taken as there be Unities in that Number; and if the Proportion be of the Greater to the Less, then shall also the quantity of the Proportion be increased by the Multiplication; but when the Proportion is of the Less to the Greater, then as the Number increaseth, the quantity of the Proportion diminisheth" (*DCo* 2.13.16; *EW* 1:164). This change was presumably due to Wallis's caustic criticism of the usage in the Latin original (*Elenchus* 24), although it is difficult to be sure of this point. In any case, the change in Hobbes's formulation of his doctrine opened a new field for verbal disputes, but these are of little interest to our investigation and can be left out of consideration here.[26]

In the final analysis, this part of the Hobbes-Wallis controversy shows that the two antagonists disagreed over fundamental questions in the philosophy of mathematics, and their disagreement is representative of unresolved tensions in the doctrine of ratios. Others had quarreled over these same issues—although perhaps without the sheer ferocity of Hobbes and Wallis—but the basic questions remained largely unresolved. It is oddly ironic that this debate should have degenerated into exactly the stale wrangling over words that so many proponents of the "new philosophy" found in the debates of their Scholastic predecessors.

4.3 THE ANGLE OF CONTACT

As their dispute progressed Hobbes and Wallis entered into a debate over the problem of the "angle of contact" between a circle and its tangent.[27] This difficulty had been the subject of controversy well before the seventeenth century, and Wallis himself published a *Geometrical Disquisition on the Angle of Contact* in 1656, independently of his

26. The changes in Hobbes's doctrine, as well as much wrangling over the question of what is to count as double or duplicate, are summarized (although not to Hobbes's advantage) by Wallis (*HHT* 70–73).

27. See Maierù 1984 and 1990 for the most comprehensive account of the problems posed by the angle of contact. Thomason 1982 contains a presentation of some of the problems engendered by the angle of contact and their alternative solutions, and Dear 1995b examines the problem in the context of Mersenne's response to Descartes's *Meditations*.

battle with Hobbes. It will be helpful to outline some of the history of this disputed issue before proceeding to an examination of the exchanges between Hobbes and Wallis.

The best place to begin is with Euclid. In the *Elements*, a plane angle is defined as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line" (*Elements* 1, def. 8). As I mentioned in chapter 3, this definition permits angles to be formed by curved and straight lines, so the three forms in figure 4.1 are angles in the Euclidean sense: The three kinds of angles here are the familiar rectilinear angle, the curvilinear angle formed by two curves, and the mixtilinear angle formed by a curve and a straight line. It is with certain mixtilinear angles that paradox first appears to threaten.

At proposition 16 of book 3 of the *Elements* Euclid proves that the angle between circle and tangent (which was also known as the "horned angle") is less than any rectilinear angle. The basic idea is as follows: We begin with a circle ABC and tangent BD (figure 4.2), and then show that any rectilinear angle $\angle DBE$ must be greater than the angle of contact $\angle CBD$. Observe that BE cuts the circle at F. But because $\angle DBF = \angle DBE$, the angle of contact must lie within the rectilinear angle $\angle DBE$, and thus $\angle CBD$ is less than $\angle DBE$. But because BE was chosen arbitrarily, the angle of contact must be less than any rectilinear angle. A similar line of reasoning establishes that the angle $\angle GBC$ (known as the "angle in a semicircle") is greater than any acute angle, prompting the suspicion that the angle in the semicircle is equal to the right angle $\angle GBD$.

The angle of contact appears paradoxical because it is a magnitude which seems to violate the principle known as the axiom of Archimedes. In one of many equivalent formulations, it appears as proposition 1 of book 10 of the *Elements* in the form of the claim:

Figure 4.1

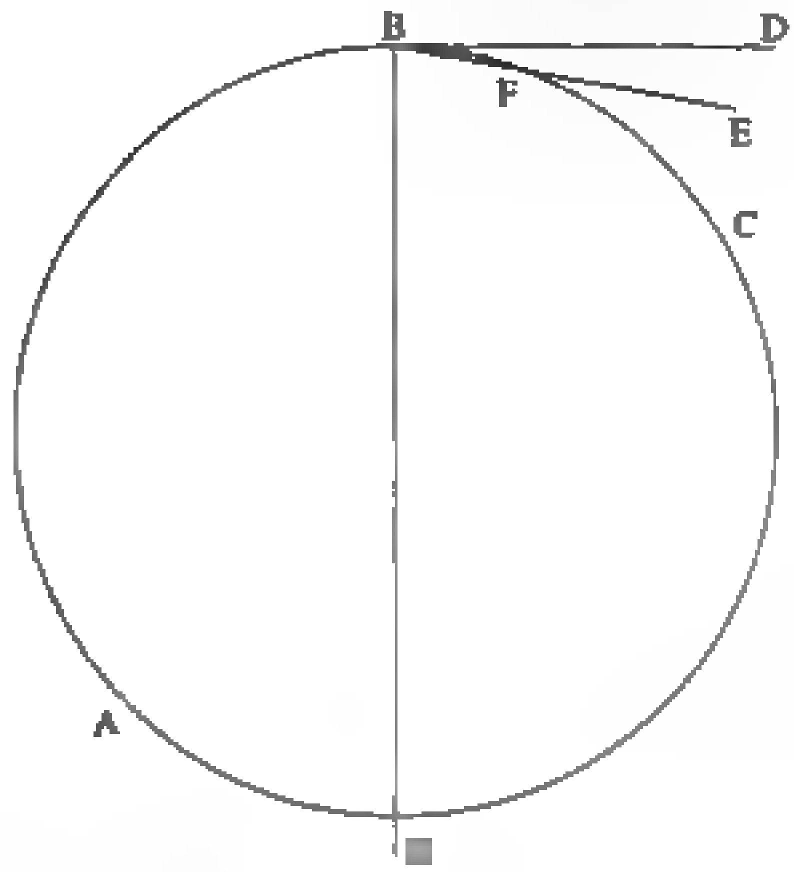

Figure 4.2

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out. (*Elements* 10, prop. 1)

The angle of contact appears to violate this condition, since no amount of division of a rectilinear angle can yield an angle less than it. However, angles of contact can be ordered by a less-than relation, because they will be greater or less depending on the radius of the circle from which they are formed. But anything compared by an ordering relation is a magnitude, so the angle of contact seems inconsistent with the Euclidean account of magnitudes.

These difficulties led commentators to speculate about the adequacy of Euclid's definition and to question whether angles were properly quantities at all. Proclus, for example, argued that no angle—whether rectilinear or mixtilinear—is a quantity:

> But if [the angle] is a magnitude and all finite homogeneous magnitudes have a ratio to one another, then all homogeneous angles, at least those in planes, will have a ratio to one another, so that a horned angle will have a ratio to a rectilinear. But all quantities that have a ratio to one another can exceed one another by being multiplied; a horned angle, then, may exceed a rectilinear, which

is impossible, for it has been proved that a horned angle is less than any rectilinear angle. (Proclus 1970, 98)

Later authors were not prepared to deny that all angles are quantities, but the attempt to fit angles into the traditional theory of magnitudes led to considerable controversy. Most notably, Clavius and the French mathematician Jacques Peletier debated the quantity of the angle of contact in a series of publications beginning with Peletier's *In Euclidis Elementa Geometrica Demonstrationum Libri Sex* (Peletier 1557).[28] In this commentary on the *Elements,* Peletier offered additions, elucidations, and corrections to the Euclidean text, claiming to have "illustrated the principles of geometry by new meditations." In particular he asserted that he had explicated "the nature, formation, and construction of the angle," which had previously not been well understood (Peletier 1557, preface sig. A2r). As part of this undertaking Peletier argued that the so-called angle of contact is not really an angle at all and that it is simply not a magnitude. Clavius opposed this argumentation in his 1574 edition of Euclid. He contended that the angle of contact is a proper angle with a magnitude, although he confessed that it is infinitely small in comparison with a rectilinear angle. The main arguments for each side of the issue are worth summarizing.

Peletier's principal argument proceeds directly from the manifest nature of magnitudes, as exemplified by the Archimedean axiom. He contends that there is "no principle in the whole of geometry that (as I may say) is more naturally true" (Peletier 1557, 75). Thus, any imaginable pair of magnitudes of the same kind must satisfy the condition that the greater of them can, by continued bisection, be made less than the other. This apparently implies that the angle of contact is not a magnitude since no amount of division will ever make a rectilinear angle less than the angle of contact. Peletier also concluded that the angle of the semicircle ($\angle GBC$ in figure 4.2) is a right angle. This follows naturally: the tangent and diameter form a right angle and the angle of the semicircle is the difference between this right angle and the angle of contact, so if the angle of contact is no magnitude, the angle of the semicircle is a right angle (Peletier 1557, 76–77).

Clavius countered this argument by claiming that the angle of contact is divisible (not by a straight line, but by a circular arc), and the classical conception of magnitudes holds that anything divisible is a

28. Maierù 1984 and 1990 examine these disputes in detail. A brief overview is also provided by Naux in his study of Clavius (1983, 325–29).

magnitude.[29] So, on Clavius's analysis, the angle of contact is a proper magnitude. He admits that the angle of contact must be smaller than any rectilinear angle, but holds that angles of contact can still be ordered by a less-than relation:

> All angles of contact whatsoever, formed by the tangent line and periphery, are less than any acute angle; but it is not necessary that they are therefore all equal to one another, for one can be greater or less than another, just as we say: any acute angle is less than any right angle, and yet these are not all equal to one another. So also every ant (to take an example from natural things) is less than any man or mountain, and still these can be truly unequal to one another. For this reason Peletier is not correct to conclude that the angle of contact is nothing. (Clavius 1612, 1:120)

Moreover, Clavius argues that the angle of a semicircle is not the same in all circles, but rather is a function of the radius of the circle. Because the angle of the semicircle is the difference between a right angle and the angle of contact, he concludes that as the radius of the circle decreases (and hence the angle of contact increases), the angle in the semicircle must also decrease.[30] But although it may decrease, the angle in the semicircle is still greater than any acute angle, as shown in proposition 16 of book 3 of Euclid.

To Peletier's claim that the angle of contact violates the Archimedean principle, Clavius replied that angles of contact and rectilinear angles are separate kinds of magnitudes, as distinct as surfaces and lines. Both parties to the dispute agreed that the Archimedean axiom applies only to magnitudes of the same kind, since the continued division of a plane figure could never produce a magnitude less than a given line. In asserting that rectilinear angles and the angle of contact are different species of magnitudes, Clavius hoped to make the Archimedean axiom irrelevant to the question of whether the angle of contact is a magnitude and thus evade Peletier's main argument. For Clav-

29. Clavius writes, "On the contrary, I say that any angle of contact can be augmented, and it can be divided infinitely by a curved line, although it cannot be divided by a right line, as Euclid correctly shows" (Clavius 1612, 1:119).

30. Clavius argues, "Although the angle of a semicircle in a greater circle is greater than the angle of a semicircle in a lesser circle, this does not mean that any angle of a semicircle is greater than a right angle. For every right angle will exceed any angle of a semicircle, by the angle of contact, which is made by the periphery and the tangent line" (Clavius 1612, 1:120).

ius, any two angles of contact will satisfy the Archimedean axiom, as will any two rectilinear angles. But because these are separate species of magnitudes, the axiom does nothing for Peletier's case.

This dispute dragged on without resolution as each party issued rebuttals to the claims of the other. Indeed, it is not obvious how the dispute could have been settled, given the conceptual resources available. Peletier and Clavius both accepted Euclid's account of angles, but the Euclidean definition is too vague to resolve the issue between them.[31] Certainly, the circumference and tangent are two lines that "lie in the same plane and meet without lying in a straight line," and there is evidently an "inclination of one to the other," so the angle of contact satisfies the Euclidean definition. But to accept such angles appears to introduce infinitesimal magnitudes into geometry, contrary to the explicitly finitistic standpoint of classical mathematics. Moreover, Clavius's account of the matter would require a substantial reworking of the Euclidean treatment of magnitudes, since declaring angles of contact and rectilinear angles two genuinely distinct species of magnitudes requires a definition of *angle* that would allow them to be distinguished. It is no wonder that the problem of the angle of contact remained unresolved into the seventeenth century.

Wallis argued for Peletier's opinion in some of his Savilian lectures on geometry, which he later collected and published in the form of a *Geometrical Disquisition on the Angle of Contact,* which appeared (along with the more famous *Arithmetica Infinitorum*) as part of the 1656 collection *Operum Mathematicorum Pars Altera* (Wallis 1656b).[32] In this treatise Wallis reviews the origin and course of the

31. Peletier comments that the Euclidean definition of angle is "well understood" and "has a tolerable and perspicuous sense." Nevertheless he prefers an alternative: "A plane angle is the cutting *[sectio]* of two lines in a plane" (Peletier 1557, 3–4). Clavius finds the Euclidean definition wholly unproblematic and rejects Peletier's alternative: "But we by no means concede to Peletier that an angle can be made only by two lines that cut one another. In order to form an angle it is sufficient that two lines in a plane incline toward one another and yet do not lie in the same right line, as is clear from the description of the plane angle taken from Euclid, even if these lines do not cut one another if produced. Of this sort is the angle formed by the periphery and the tangent, which is a true angle, as we have shown above" (Clavius 1612, 1:120). The reason for this disagreement should be clear: since the tangent and periphery of the circle touch without cutting, Peletier's alternative definition guarantees that the angle of contact is not an angle at all.

32. The provenance of the *Geometrical Disquisition on the Angle of Contact* as Savilian lectures is established by Wallis's remark that he had earlier undertaken "to show by illuminating demonstrations that truth stood on the side of Peletier in my public lectures, which I then gave over to the printer" (*Elenchus* 34).

controversy, undertaking to prove that "the angle of contact is a nonangle, or a nonquantum; and thus the angle in the semicircle is equal to a rectilinear right angle" (*Angle of Contact* 15; *OM* 2:630). The crux of Wallis's case is his argument that all angles, whether rectilinear or mixtilinear, are homogeneous. Establishing such a claim would clearly render Clavius's account of the angle of contact untenable. However, before undertaking to prove this, Wallis introduces a crucial assumption concerning the definition of angles. Where Euclid had defined angles in terms of lines that incline toward one another, Wallis suggests that the concept of "inclination" be so understood that only those lines that actually cut have a mutual inclination toward one another.[33] The proof of the homogeneity of all angles then proceeds, although the argumentation is not decisive. The homogeneity of all rectilinear angles is entirely trivial and undisputed, and Wallis thinks that the same can be said for curvilinear angles:

> And further, to any possible rectilinear angle it is also possible to assign a curvilinear angle equal to it: just as Clavius himself demonstrates out of Proclus, at the fifth definition of the fifth book of the *Elements*.[34] For example, the curvilinear angle *DAE* [in figure 4.3] is equal to the rectilinear angle *BAC*; and in the same way to any assignable rectilinear angle there is assignable an equal curvilinear angle; and thus curvilinear angles can bear any given ratio not merely to one another, but also to any rectilinear angle, which ratio rectilinear angles themselves can bear to one another. (*Angle of Contact* 5; *OM* 2:612)

The weakness of this argument is that it applies only to curvilinear angles where the intersecting lines cut one another. It is certainly true

33. Wallis declares: "Peletier preferred this [Euclidean] definition of angle to be slightly altered, so that it is understood of the concourse of lines that cut one another. But no necessity forces this kind of change to be made. Although it is to be said, with Peletier, that only those lines contain an angle that (if produced) will cut one another, nevertheless it is no less true that, according to the same principles, only those lines meet that are inclined to one another. Which shows that the circumference does not make an angle with the tangent right line, and which also will show that these lines are not inclined to one another. So there is no need to change the Euclidean definition on account of this opinion" (*Angle of Contact* 3; *OM* 2:607).

34. The reference here is to Proclus's discussion of the fourth postulate of Euclid, which asserts that all right angles are equal to one another. Proclus considers the case of a right angle formed by the diameters of two circles, and argues that the angle formed by the intersecting circular peripheries will be equal to a right angle (Proclus 1970, 148–50). Clavius considers the same issue in his remarks on the Euclidean definition of proportionality in book five of the *Elements* (Clavius 1612, 1:208).

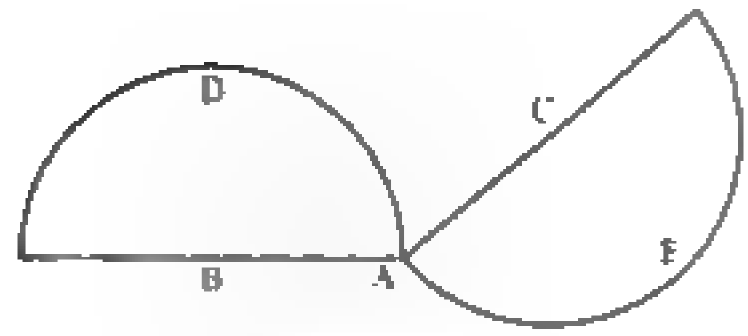

Figure 4.3

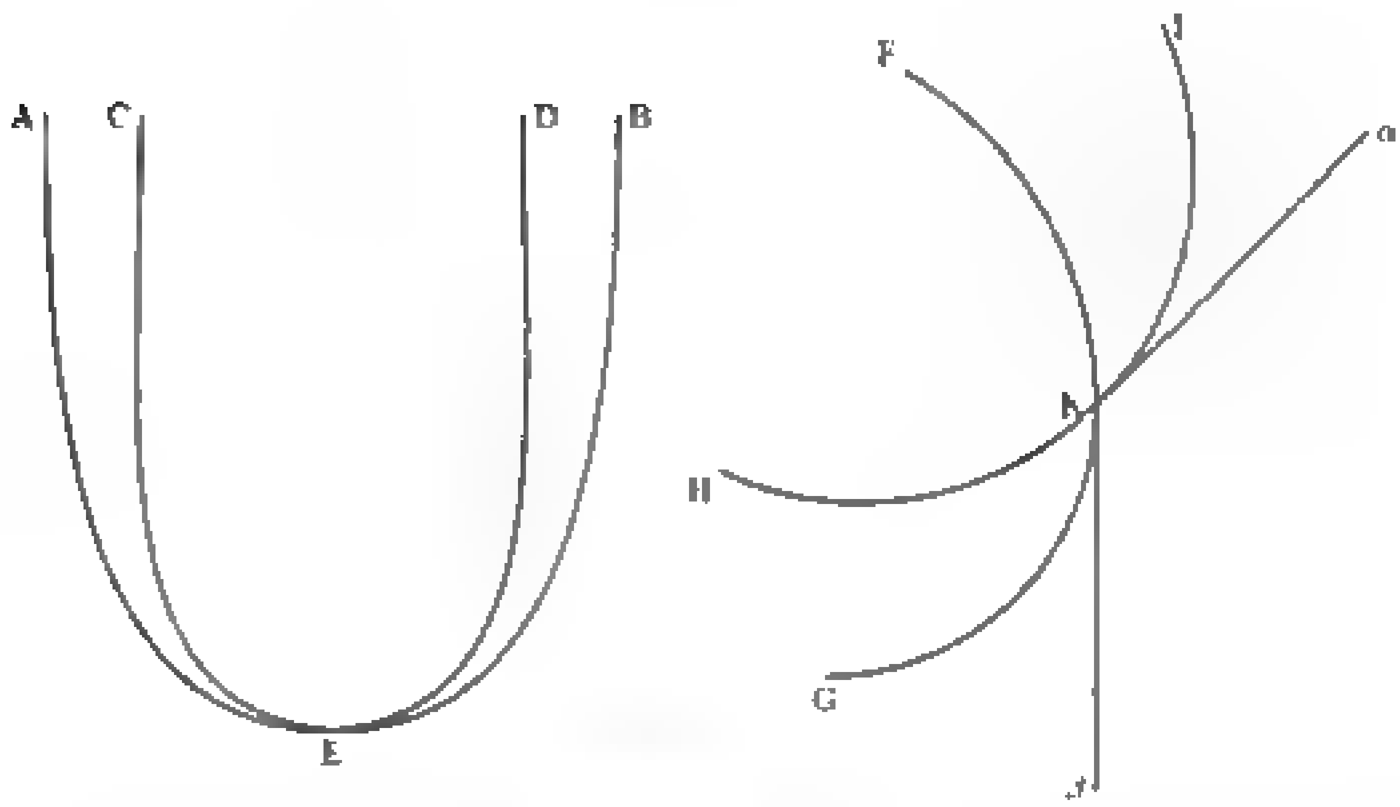

Figure 4.4

that, to any rectilinear angle, there will correspond any number of curvilinear angles (namely, those formed by curves tangent to the legs of the angle at their point of intersection). But this does not show that all curvilinear angles are homogeneous, unless it could also be shown that for every curvilinear angle there is a rectilinear angle equal to it. As it turns out, no rectilinear angle corresponds to curves that touch one another without cutting. In figure 4.4, for example, the curvilinear arcs *AB* and *CD* have a common point at *E,* but no rectilinear angle can be assigned equal to the angle of contact *BED.* Any two curves that cut one another will form angles homogeneous to rectilinear angles; thus the curves *FG* and *HI* in figure 4.4 cut at the point *K,* and the angle *GKI* is equal to the rectilinear angle $\alpha K\beta$ formed by the tangents $K\alpha$ and $K\beta$.

Nevertheless, Wallis's case is not decisive in showing that all curvilinear angles are homogeneous, unless he can provide an independent reason for restricting the class of curvilinear angles to the case of curves that cut one another. Having satisfied himself that he has shown all rectilinear and curvilinear angles to be homogeneous, Wallis then

argues that any mixtilinear angle will likewise be homogeneous to any rectilinear angle. But again, he restricts his attention to cases where a right line cuts a curve. He argues:

> But a mixed angle can also have a ratio to a rectilinear angle, which I think Clavius would not deny. For a right line, if not one that is tangent, then at least one that cuts the periphery (both inside and out) will make a mixed angle with the periphery, which will be greater than at least some rectilinear angle, which then can be multiplied and exceed some assigned rectilinear angle (and on the other hand some rectilinear angle can exceed it). And so, as Clavius himself confesses, it must be said that there is a ratio, in accordance with definition 4 of book 5 of Euclid. (*Angle of Contact* 6; *OM* 2:613)

This argumentation clearly depends upon the requirement that an angle is formed only by two lines that cut one another. As such, it can do the work Wallis intends for it only if Euclid's definition of angle is taken to imply that when two lines incline toward each other they must cut one another. Wallis was sensitive to this difficulty and tried to overcome it by arguing that lines that are tangent but do not cut (such as those forming the angle of contact) actually do not incline toward each other. Although such lines may *seem* to have mutual inclination in the neighborhood of their concurrence, they do not incline at the point of contact.[35]

Hobbes took a quite different view of the matter. As we have seen, he distinguished between angles of circumlation (or "angles simply so called") from angles of contingence, basing this distinction on "the two ways by which two lines may diverge from one another." Angles of the former kind are generated by the rotation of a line about one of its endpoints, while the latter arise from "continual flexion or curvation in every imaginable point" (*DCo* 2.14.7; *OL* 1:160). This difference in origin requires that the angles be measured differently. Al-

35. Wallis writes, "When Clavius says, arguing from the definition of plane angle, '[t]hat in order to make a plane angle it suffices that two lines in a plane incline to one another,' but it is not required 'that they cut one another,' I admit that this is so; but I deny that the periphery and tangent right line 'are inclined to one another,' at least as far as the point of concourse; they coincide, but they are not inclined. I acknowledge also that (as he elsewhere says) '[t]he angle consists in the inclination of lines at a point that do not lie in a straight line'; but I do not acknowledge that these lines *are inclined*. Rather, with Peletier I affirm that only those lines *incline* (in the point of concourse) that, if produced, will cut one another: nor does Clavius ever show anything to the contrary" (*Angle of Contact* 6; *OM* 2:614).

though both kinds of angle are magnitudes (since they can be greater or less), they are heterogeneous and cannot be compared with one another as to quantity. Angles of circumlation are measured by the ratio of a comprehended arc to the circumference of a circle; angles of contact are measured in terms of the degree of curvature or flexion with which they are produced.[36]

Hobbes's solution to the problem of the angle of contact thus allows such angles to be construed as magnitudes, but magnitudes heterogeneous to rectilinear angles and measured differently. As he sums up the matter in *De Corpore:*

> An angle of contact, if it is compared with an angle simply so called, no matter how small, stands to it in a ratio of a point to a line; that is, no ratio at all, nor any relation of quantity. For . . . an angle of contact is made by continual flexion: so that in the generation of it there is no circular motion at all, in which consists the nature of an angle simply so called; and therefore it cannot be compared with it according to quantity. (*DCo* 2.14.16; *OL* 1:169–70)

Although Hobbes tended to side with Clavius in this matter, he was not entirely opposed to Peletier's account. In particular, he faulted Clavius for claiming that the angle in the semicircle depends upon the radius of the circle in which it is formed. On Hobbes's interpretation of the matter, the angle in the semicircle is an angle of circumlation formed by the divarication of the radius from the periphery of the circle. As such, it is a right angle that must be measured by considering the radius and tangent. Thus Peletier's opinion can be partially vindicated. Or, to put the matter negatively, as Hobbes does in the *Six Lessons,* he disagrees with both authors and with Euclid himself: "[I]n this same Question, I am of opinion that *Peletarius* did not well in denying the Angle of Contingence to be an Angle. And that *Clavius* did not well to say, *the Angle of a Semicircle* was less than *a Right-*

36. It is worth noting that this was not always Hobbes's opinion. In his critique of Thomas White's *De Mundo,* Hobbes devotes chapter 23 to a discussion of the angle of contact. His conclusion is that "[i]t is doubtful what measure should be assigned to the angle of contact. If, however, this is an angle and is quantity, then it must possess a measure of the same kind as itself; whence arose the argument between Clavius and Peletier. Here it seems to me that . . . the opinion of Peletier, though true, is [based on] ineffectual arguments; but Clavius, besides putting forward an erroneous view, has contrived further mistaken tenets in order to lend it support" (Hobbes 1976, 262).

lined Angle. And that *Euclide* did not well to leave it so obscure what he meant by *Inclination* in the Definition of a *Plain Angle*" (*SL* 3; *EW* 7:258).

Hobbes's solution to this problem is quite ingenious. By distinguishing two kinds of angles, Hobbes paves the way for separate means of measuring such angles. He thus defends the opinion that the angle of contact is a quantity by showing how to measure it, but in distinguishing the measure of the angle of contact from the measure of more familiar angles he avoids the paradox that threatens if we allow the magnitude of the angle of contact to be compared with that of an ordinary rectilinear angle.

Hobbes had no patience with Wallis's approach to this difficulty. In the third of his *Six Lessons* Hobbes mounted a counterattack against the criticisms of Wallis by reviewing the principal arguments in the *Geometrical Disquisition on the Angle of Contact*. He notes the shortcomings in Wallis's argument for the homogeneity of all angles,[37] and finds fault with his doctrine that there is no mutual inclination between circle and tangent at the point of contact. This latter doctrine poses particularly difficult problems for Wallis, because he is committed to the unusual notion that a curve and its tangent have no inclination at their common point, while a curve and its secant do manifest such an inclination. The idea that points can have inclination to one another is certainly not one with a great deal of intuitive appeal, especially given Wallis's understanding of points as lacking dimension. Hobbes ridicules the doctrine with the remark "I pray you tell me what straddling there is of two coincident Points, especially such Points as you say are nothing. When did you ever see two nothings straddle?" (*SL* 3; *EW* 7:263). Hobbes even puts forward an argument designed to show that lines that cut one another need not form an angle homogeneous to a rectilinear angle.[38] The argument is worth considering briefly for the purposes of clarifying some of the issues in dispute. Hobbes writes:

37. Thus, Hobbes notes, "Your first argument therefore is nothing worth, except you make good that which in your second Argument you affirm, namely, That all Plain Angles, not excepting the Angle of Contact, are (*Homogeneous*) of the same kind. You prove it well enough of other Curvilineall Angles; but when you should prove the same of an Angle of Contact, you have nothing to say but . . . *whence should arise that diversity of kind, which [Clavius] dreams of, neither can he at all shew, nor I dream*" (*SL* 3; *EW* 7:259).

38. This example, I must confess, undermines my earlier account of Hobbes's distinction between angles of contingence and angles of circumlation in Jesseph 1993a, where I claim that the two classes of angles are distinguished by whether or not the lines

And why have two straight Lines Inclination before they come to touch, more then a straight Line and an Arch of a Circle? And in the Point of Contact it self, how can it be that there is less Inclination of the two Points of a straight Line and an Arch of a Circle, then of the Points of two straight Lines? But the straight Lines you say will cut; Which is nothing to the Question; and yet this also is not so evident, but that it may receive an objection. Suppose two Circles AGB and CFB to touch in B [figure 4.5], and have a common Tangent through B. Is not the Line CFBGA a crooked line? And is it not cut by the common Tangent DBE? What is the Quantity of the two Angles FBE and GBD, seeing you say neither DBG nor EBF is an Angle? 'Tis not, therefore, the cutting of a crooked Line, and the touching of it, that distinguisheth an Angle simply, from an Angle of Contact. That which makes them differ, and in kind, is, that the one is the Quantity of a *Revolution*, and the other the Quantity of *Flexion*. (*SL* 3; *EW* 7:260–61)

This argument is intriguing for several reasons. First, it highlights Hobbes's methodological tenet that the different species of geometric objects must be distinguished by their causal origin. The angle of contact is formed by the flexion of a line, while ordinary angles are formed by the rotation of lines; thus, the measure of the two kinds of quantity must be adapted to the different manners in which the quantities are produced. Wallis follows the more traditional method of arguing from a definition that is supposed to explicate the essence of the thing defined without reference to the manner in which it is produced. He thus defines the angle in terms of the mutual inclination of two lines, so that the question of whether the circle and its tangent fall under the relevant definition reduces to the question of whether they are inclined to one another. Wallis further sharpens his definition by requiring that lines inclined to one another must also cut, and he takes this criterion to solve the problem. In contrast, Hobbes takes the question of whether two lines cut to be entirely irrelevant to the manner in which they are produced, and hence useless in resolving the issue of the angle of contact.

Beyond this, it should be noted that the example Hobbes gives does

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that form them cut one another. As the following example shows, this is not a completely accurate representation of Hobbes's views, since he constructs a curve that he claims exemplifies the angle of contact, even though the two lines in it cut one another.

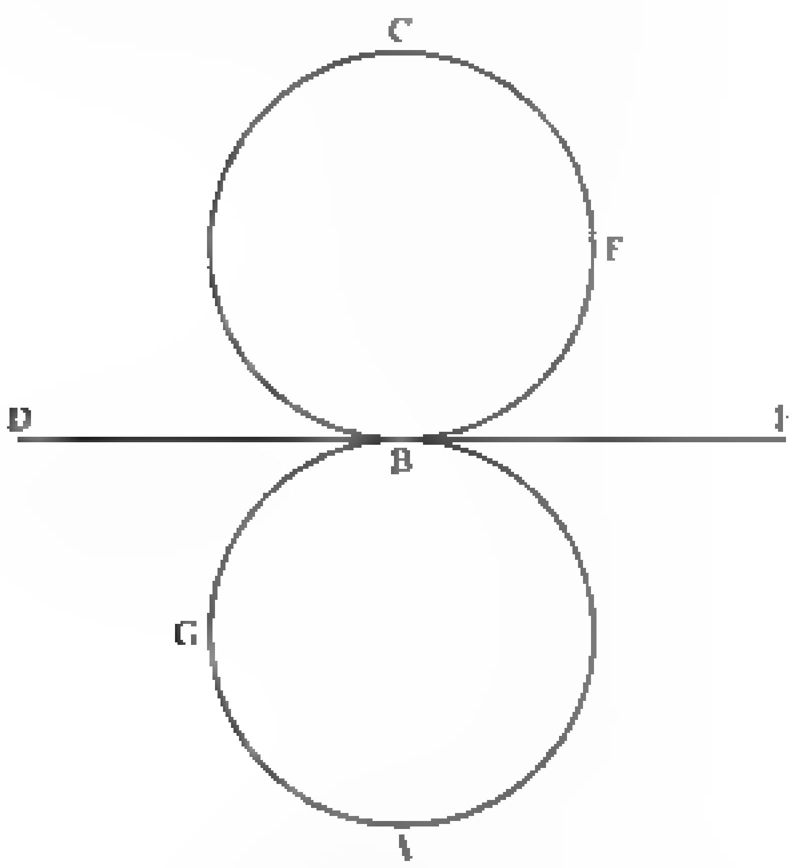

Figure 4.5

pose something of a problem for Wallis: the lines *CFBGA* and *DBE* do, in fact, cut one another, but because *DBE* is also the tangent to the curve at *B*, they do so without forming an angle (or, at least, without forming an angle in the sense that Wallis is prepared to recognize). Thus, Wallis must accept that the angles *DBG* and *FBE* have no magnitude. The case shows that the criterion for formation of an angle must be more complex than simply requiring that two curves cut. To vindicate his account, Wallis must add the requirement that the lines forming an angle cut without being tangent; in those cases where a tangent cuts the curve (such as points of inflexion, as in Hobbes's counterexample), there is no angle.

Hobbes and Wallis traded polemics over the angle of contact for many years, but the fundamental positions did not vary in the least. Hobbes continued to accuse Wallis of misunderstanding the nature of angles; Wallis countered that Hobbes had granted him every contested point. Ultimately, Wallis seems to have been intent upon a willful misunderstanding of Hobbes's solution to the problem, as is evident in his reply to Hobbes's *De Principiis et Ratiocinatione Geometrarum:*

> What he saith here . . . concerning the *Angle of Contact;* amounts but to thus much, That, by the *Angle of Contact,* he doth not mean either what *Euclide* calls an *Angle,* or any thing of that kind; (and therefore says nothing to the purpose of what was in controversie between *Clavius* and *Peletarius,* when he says, that *An Angle of Contact hath some magnitude:*) But, that by the *Angle of Contact,* he understands the *Crookedness of the*

> *Arch;* and in saying, *the Angle of Contact hath some magnitude,* his meaning is, that the *Arch of a Circle hath some crookedness,* or is a *crooked line:* and that, of equal Arches, That is the more crooked, whose chord is shortest: which I think none will deny. . . . But, why the *Crookedness of an Arch,* should be called *an Angle of Contact;* I know no other reason, but, because Mr. *Hobs* loves to call that *Chalk,* which others call *Cheese.* (Wallis 1666, 292)

The objection seems quite beside the point. As noted, Hobbes based his distinction between two types of angles on straightforward geometric criteria and introduced two methods for measuring these distinct kinds of magnitudes. Wallis's comments here are revealing because they show that he stubbornly refused to accept Hobbes's case for distinguishing the measure of the angle of contact from the measure of ordinary angles.

Curiously, Wallis eventually adopted a position not radically different from that of Hobbes, although it emerged only in 1684, some five years after Hobbes's death. In his *Defense of the Treatise of the Angle of Contact* Wallis defended his previous treatise on the subject and added some elucidations and clarifications. In the fourth chapter he took up the task of explaining away the apparent "Paradox to sense" that a curve can divaricate from a line without forming an angle. He does this by introducing a distinction between flexion and fraction as the two means by which one line may depart from another:

> A streight-line as APp, which we may suppose in a Perpendicular position to AC, may come to change its position, as from Perpendicular to Parallel, (as to some part of it) either by a Break, as at B; or by more such, as at D[,] E; making so many Angles, as there are Breaks; (each part retaining its own streightness as before) or (without any Break) by one continued Bowing, as AF. (1684, 89)

In the former case, the lines are removed from each other by fraction or breaking; in the latter case by flexion or bowing. Fraction produces a genuine angle, while the uniform flexion at every point leaves no angle. It is obvious that the distinction here is the same as Hobbes's distinction between angles of circumlation and angles of contingence. Although Wallis would never have acknowledged it, it is amusing to see that his treatment of the angle of contact ultimately bears a strong resemblance to that of his most bitter antagonist.

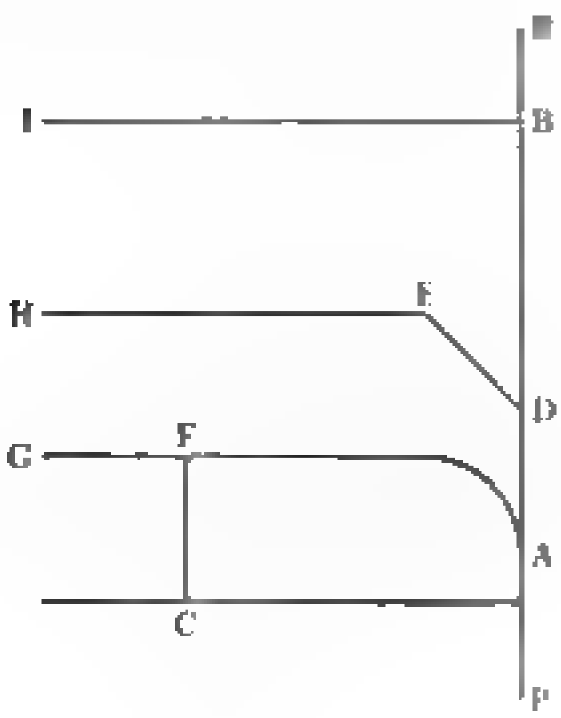

Figure 4.6

4.4 HOBBES, WALLIS, AND THE INFINITE

The topic of the infinite—and particularly the infinitely small—was a source of debate throughout the seventeenth century, and it is no surprise that Hobbes and Wallis should clash over it as well.[39] Their disagreement in this matter focused on Wallis's use of the infinite in his version of the method of indivisibles. The fundamental question at issue is whether a properly developed mathematical theory may postulate the existence of infinitesimal magnitudes, that is to say quantities greater than nothing but less than any assignable finite magnitude.

We saw in chapter 1 that Cavalieri's method of indivisibles was an important tool for the investigation of quadratures, and the method is often portrayed as an example of infinitesimal mathematics. However, Cavalieri himself avoided any direct commitment to the reality of infinitesimal magnitudes. Rather than taking geometric magnitudes as composed of infinite collections of infinitely small parts, Cavalieri tried to fit indivisibles into the classical theory of magnitudes by proposing that "all the lines" of a figure could be a new species of magnitude, on a par with lines, angles, surfaces, and solids. Thus, rather than assert that the area of the figure F is composed of an infinite sum of lines l_i, Cavalieri claims that the collection of lines enclosed in the figure F_1 stands in a determinate ratio to the collection of lines enclosed in another figure F_2. This approach is silent on the issue of the reality of infinitesimal quantities and requires only that one acknowledge that

39. This part of the dispute has been examined by Mancosu and Vailati (1991, 64–70) and Beeley (1996, 272–74).

"all the lines" contained in figures are themselves magnitudes that can be compared to one another in ratios. Later mathematicians were not necessarily as cautious or scrupulous in their approach to the infinite.

Wallis was an enthusiastic proponent of the infinitesimal, and in his presentation of the method he made no scruple of speaking of continuous geometric magnitudes as literally composed of infinite collections of infinitely small parts. Where Cavalieri had tried to avoid any commitment to the reality of infinitesimal magnitudes, Wallis saw no difficulty in taking a line or curve as an aggregate of indivisible points, a surface as an infinite collection of lines, and so forth. He was also quite happy to assume that Cavalieri's presentation of the method had resolved any lingering foundational doubts about the infinite. In his account of the foundation of the method:

> It is understood that any continuum (according to the Geometry of Indivisibles of Cavalieri) consists of an infinite number of indivisibles.
>
> As from an infinity of points, a line; a surface from an infinity of lines; and a solid from an infinite number of surfaces; so also time from an infinity of temporal moments, etc. That is . . . from homogeneous particles, infinitely small and infinite in number, and . . . equal in at least one dimension. (*Mechanica* 5; *OM* 1:645–46)

Notwithstanding the novelty of this approach, at least when judged by the classical standards of rigor, Wallis insists that it is merely a notational variant of the classical "method of exhaustion," which eschews infinitesimal considerations and proceeds by constructing a sequence of approximations converging to the desired result. In Wallis's estimation:

> The Method of Exhaustions (by Inscribing and Circumscribing Figures, till their difference becomes less than any assignable,) is a little disguised, in (what hath been called) Geometria Indivisibilium . . . which is not, as to the substance of it, really different from the Method of Exhaustions, (used both by Ancients and Moderns,) but grounded on it, and demonstrable by it: But is only a shorter way of expressing the same notion in other terms. (*Treatise of Algebra* 74, 285)

Not surprisingly, Wallis admits that the new method must be "applied with due caution" (*Mechanica* 5; *OM* 1:646) in order to avoid familiar

paradoxes involving the infinite and the composition of the continuum.

Wallis's procedure rests (in part) on his conviction that arithmetic is the genuine foundation of all mathematics, and that geometry is a science subordinate to that of arithmetic. The true method of solving geometrical problems, according to Wallis, requires that they be represented arithmetically as infinite sums and their solution sought at the level of arithmetical calculation. He portrays this approach as "much more gra[c]eful and agreeable, than the Operose Apagogical Demonstrations . . . which some seem to affect," largely because there is no need for complex reductio ad absurdum arguments in the style of the classical method of exhaustion. Such classical techniques were made necessary "for reasons which now (in great measure) are ceased since the introducing the Numerical Figures, and (much more) since the way of Specious Arithmetic" (*Treatise of Algebra* 78, 298).

Another great advantage to be gained by the new methods is that they show quadrature results to be independent of any specifically geometrical content. The use of infinite series in solving geometric quadratures avoids dependence upon diagrams and represents the results in a manner independent of our intuitions concerning the structure of geometric space. In Wallis's defense of his methods, the lack of diagrams becomes a peculiar strength of his approach:

> If any think [these demonstrations] less valuable, because not set forth with the Pompous ostentation of Lines and Figures: I am quite of another mind. For though such Lines and Figures be necessary where the Truth of a Proposition depends on Local Position: And though they be otherwise of use, sometimes for assisting the Fansy or Imagination (shewing that to the eye, by way of instance, in one particular case, as that of Lines; which is abstractly true in all kinds of Quantity whatever:) Yet where the truth of the Proposition depends merely on the nature of Number or Proportion . . . It is much more natural to prove it abstractly from the nature of Number and Proportion; without such embarrassing the Demonstration. (*Treatise of Algebra* 78, 298)

Wallis characterizes this approach to geometry as a kind of "*Abstractio Mathematica* . . . whereby we separate what is the proper Subject of Inquiry, and upon which the Process proceeds, from the impertinences of the matter (accidental to it,) appertaining to the present case or particular construction" (*Treatise of Algebra* 76, 292). This perspective is evident even in the title of Wallis's main work on the method

of indivisibles—*Arithmetica Infinitorum*, or the arithmetic of infinities. His point is that the infinite, and especially the infinitely small, falls within the purview of the general science of quantity, so that the arithmetic of infinities is a legitimate branch of mathematics useful for the investigation of quadratures.

As I mentioned in chapter 1, Wallis's arithmetic of infinities approaches geometric problems of quadrature by first representing a figure as an infinite sum of infinitesimal elements and then calculating the appropriate sum in order to find the area of the figure. This approach requires the evaluation of infinite sums in order to obtain a quadrature. Fundamental to this process is Wallis's method of "induction," which infers the value of an infinite sum from an investigation of the initial cases. As he expresses it:

> The simplest way of investigating this and other problems is to set forth a certain number of cases and observe the resulting ratios, and then compare them with one another in order that the universal proposition can then be known by induction. For example, we have
>
> $$\frac{0+1}{1+1} = \frac{1}{2}$$
>
> $$\frac{0+1+2}{2+2+2} = \frac{3}{6} = \frac{1}{2}$$
>
> $$\frac{0+1+2+3}{3+3+3+3} = \frac{6}{12} = \frac{1}{2}$$
>
> $$\frac{0+1+2+3+4}{4+4+4+4+4} = \frac{10}{20} = \frac{1}{2}$$
>
> $$\frac{0+1+2+3+4+5}{5+5+5+5+5+5} = \frac{15}{30} = \frac{1}{2}$$
>
> $$\frac{0+1+2+3+4+5+6}{6+6+6+6+6+6+6} = \frac{21}{42} = \frac{1}{2}$$
>
> And in the same way, however far we may proceed, this will always produce a ratio of 1 to 2. (*AI* 1; *OM* 1:365)

The difficulty with such a procedure is obvious enough, since it in effect draws a conclusion about the infinite case from the examination of a few initial cases. In the example cited the ratio of quantities remains constant as the number of terms increases, but Wallis also sought to determine ratios between infinite series where the ratios vary but tend

toward a fixed value. In both kinds of reasoning there remains the difficulty of trying to assign a finite value to an infinite sum in the absence of principles that extend to the infinite case. The development of the calculus in the eighteenth and nineteenth centuries eventually led to a rigorous theory of infinite series, but in Wallis's day there were no clearly defined criteria for the summation of infinite series. The rather casual manner in which Wallis approached questions of the infinite has been widely acknowledged, and we need not dwell on it except to note that it was an inviting target for Hobbes's criticism.[40]

Hobbes wasted no time in attacking this weak link in Wallis's mathematics. He ridiculed such "inductions" in his *Six Lessons* with the words: "Egregious Logicians and Geometricians, that think an *Induction* without a *Numeration* of all the particulars sufficient to infer a Conclusion universal, and fit to be received for a Geometricall Demonstration!" (*SL* 5; *EW* 7:308). Lacking a coherent theory of the infinite or a methodology that could rigorously establish results for the evaluation of infinite sums, Wallis tried to defend himself by insisting that his methods comport with the classical standards. In particular, he argued that by including the expression "and the like in other cases," the reasoning "may passe for a proofe, till there be a possibility of giving some instance to the contrary; which, here, you will never be able to doe" (*Due Correction* 42). The alleged classical warrant for this procedure is Euclid's procedure of effecting a construction for a single case, say a given line designated $\alpha\beta$, and then generalizing the result to any other line.[41] The weakness of this pretense is manifest: when a construction

40. Scott 1938, 19–21, is representative in regarding Wallis's approach to the infinite as an untoward lapse that can still be interpreted charitably: "Unfortunately, however, he falls into the not uncommon error of regarding infinitely small and zero as synonymous expressions, *i.e.* as though $a - a$ were the same thing as $1/\infty$. . . . Frequently he treats infinity as though the ordinary rules of arithmetic could be applied to it. . . . But this is perhaps understandable. For many years to come the greatest confusion regarding these terms persisted, and even in the next century they continued to be used in what appears to us as an amazingly reckless fashion." Prag 1931, 390, observes that Wallis himself did not use the term "proof" (or its Latin equivalents) in speaking of his methods or their results; this suggests that perhaps the Savilian professor himself did not regard his procedures as completely demonstrative.

41. As Wallis puts it:

> [I]f such an induction may not passe for proofe, there is never a proposition in Euclide demonstrated. For all along he takes no other course then such, (or at least grounds his Demonstrations on propositions no otherwise demonstrated.) As for instance; he proposeth it in general . . . *to make an Equilater triangle on a line given.* And then shews you how to doe it upon the line $\alpha\beta$ which he then shews you: and leaves you to supply, *and the same by the like meanes, may be*

is effected on a line or figure "in general position," the construction does not depend upon any specific features of such lines and figures, and hence the result may be generalized to other lines and figures that share the properties appealed to in the construction. Wallis's results for the summation of infinite series lack this kind of generality. Indeed, without an analogue of the schema for mathematical induction, Wallis's results are not rigorously demonstrated. In this particular case Hobbes can embarrass Wallis, precisely because there was no adequate theory of convergence upon which to base the arithmetic of infinities. In point of fact, if the "official" standard of rigor from the seventeenth century is applied to Wallis's principal theorems they must be accorded the status of plausible conjectures rather than genuine demonstrations. Hobbes, of course, never tired of insisting that Wallis's results were undemonstrated, and his continued polemics against the *Arithmetica Infinitorum* are a recurring theme in the dispute.

I showed in chapter 3 that Hobbes was influenced by Cavalieri's method of indivisibles and that he adopted it almost without alteration. Such obvious acceptance of the method might make it surprising that Hobbes should attack Wallis's use of indivisibles, but it is clear that Hobbes understood the method in completely different terms than Wallis. In fact, Hobbes saw Cavalieri as working with a conception of points and lines that differs very little from his own materialistic program for the foundations of mathematics. In particular, Hobbes thought that Cavalieri attributed breadth to lines, and he took this to be part of the foundation of the method of indivisibles. This is brought out in the part of the *Examinatio* that deals with the concept of homogeneous magnitudes, where one of the interlocutors in Hobbes's dialogue declares that:

> Those things that can exceed one another when multiplied are homogeneous, and these are measurable by a measure of the same kind, as lines are measurable by lines, surfaces by surfaces, and solids by solids. However, things heterogeneous are measured by different kinds of measures. Nevertheless, if lines are considered as the most minute parallelograms, as they are considered by those who use the method of demonstration that Bonaventura Cavalieri calls the doctrine of *Indivisibles,* then there will be a ratio between a *right line* and a *plane surface.* And indeed

done upon any other streight line; and then inferres his generall conclusion. Yet I have not heard any man object, that the induction was not sufficient, because he did not actually performe it in all lines possible. (*Due Correction* 42)

> such lines, when multiplied, can exceed any given finite plane surface. (*Examinatio* 2; *OL* 4:75)

It is likely that Cavalieri's deliberate vagueness about the foundations of his method is responsible for Hobbes's unusual interpretation of the method. There is also the possibility that Cavalieri encouraged such an interpretation with some of his less strict statements of the method of indivisibles. In particular, such expressions as the "cloth and book" metaphor in the *Exercitationes* could easily have encouraged Hobbes's misinterpretation. There, Cavalieri writes "it is clear that plane figures are to be conceived by us as like cloth made up of parallel threads, and also solids as like books composed out of parallel pages" (Cavalieri 1647, 3). Despite its apparent affinity to Hobbes's conception of mathematical objects, this passage is really quite misleading. Perhaps not surprisingly, the metaphor was widely known and used to justify various interpretations of Cavalieri's methods.[42] In fact, Cavalieri immediately adds an important qualification: material objects such as cloth and books are composed out of finite collections of finite parts, where each part has some thickness, while the lines and planes considered in the method of indivisibles are "indefinite" in number and are conceived as lacking thickness (Cavalieri 1647, 4). Whatever the source of his interpretation of Cavalieri, Hobbes seems to have seen in him one of his few mathematical allies, notwithstanding the fact that Cavalieri opposed Hobbes's doctrine of the fundamentally material nature of mathematical bodies.[43] Since there is no deep doctrinal affinity between these two thinkers, the latter's reticence on foundational issues

42. De Gandt 1991, 161 n. 12, observes that this passage is not representative of Cavalieri's doctrine, but "unfortunately [it] was more widely known and quoted than his rigorous and technical formulations."

43. The divergence between Hobbes's and Cavalieri's conceptions of mathematics can be understood from Cavalieri's comments on the relationship between mathematics and the study of nature. Cavalieri declares: "Not all things that are considered by geometers are necessarily found in the nature of things just exactly [as they are in mathematical theories], for natural things, frequently sullied with matter and enmeshed in imperfections, are so inconstant that they can hardly be the object of science. For this reason mathematicians are constrained to abstract from this sensible matter and to pursue their studies in the purest and simplest figures, which always remain the same. And thus Archimedes presented the doctrine of spirals, even if perhaps no line of this sort is found in nature. Thus Galileo, and more recently Torricelli, examined the effects produced by local motion according to the same suppositions, regardless how nature in itself may work by local motion. All of which things, even if they were only minimally employed in human affairs, are by no means to be neglected, as they serve as nourishment for our contemplative powers" (Cavalieri 1647, 322). This doctrine is clearly at odds with the Hobbesian conception of mathematics.

is responsible for Hobbes's approval (and reinterpretation) of Cavalieri's doctrines.[44]

Whatever the ultimate source of Hobbes's interpretation of Cavalieri, he insisted that the method of indivisibles could not be properly comprehended without attributing extension to points and breadth to lines. Thus, in the fifth of his *Six Lessons,* he remarks to Wallis: "[Y]ou think it will pass for current, without proof, that a Point is nothing. Which if it do, Geometry also shall pass for nothing, as having no ground nor beginning but in nothing. But I have already in a former Lesson sufficiently shew'd you the consequence of that opinion. To which I may add, that it destroys the method of *Indivisibles,* invented by *Bonaventura;* and upon which, not well understood, you have grounded all your scurvy book of *Arithmetica Infinitorum*" (*SL* 5; *EW* 7:300–301).

Hobbes may well be wrong in thinking that Wallis's methods depend upon the supposition that a point is "nothing," but his complaint can be rephrased as a more cogent objection to the doctrine of infinitesimal parts. If, indeed, a line is "breadthless length," then the addition of lines can never constitute a breadth because it would simply be a sum of zeroes. Wallis himself sometimes preferred to speak of surfaces as composed of infinitely narrow parallelograms rather than breadthless lines, but this minor change does nothing to make his doctrine more coherent. In his *Treatise of Conic Sections* he announces that "I suppose, to begin with (according to the *Geometry of Indivisibles* of Bonaventura Cavalieri), any plane to be made up *[conflari]* so to speak out of an infinity of parallel lines; or (which I prefer) from an infinity of parallelograms of the same altitude. Let the altitude of any one of them be $1/\infty$ of the whole, or an aliquot part infinitely small (the sign ∞ denoting an infinite number), and the altitude of all together being equal to the altitude of the figure" (*Conic Sections* 1; *OM* 1:297). Wallis makes no attempt to distinguish an infinitely narrow parallelogram from a line, except to say that the former is supposed to "be dilatable, or to have so much thickness that by infinite multiplica-

44. In this context it is interesting to contrast Hobbes's evaluation of Cavalieri (and his associate Evangelista Torricelli) with the work of later mathematicians, particularly Wallis. In the *Admonitio ad Lectores* at the end of the *Lux Mathematica,* Hobbes remarks that "we had, avid reader, very skillful masters of the human sciences (I speak of geometry and physics) in the most distant ages: above all in geometry Euclid, Archimedes, Apollonius, Pappus, and others from ancient Greece. More recently we have Cavalieri and Torricelli from Italy. . . . But today, I say we are not even staying even, but instead are falling backward" (*Lux* 14; *OL* 5:147–48). Hobbes presumably intends Wallis as one of the principal examples of declining standards.

tion it can acquire a certain altitude or latitude, namely as much as is in the altitude of a figure" (*Conic Sections* 1; *OM* 1:297). However, this alleged difference is immediately rendered worthless when Wallis announces that the altitude of the parallelograms that compose a surface "is supposed to be infinitely small, that is, no altitude, for a quantity infinitely small is not quantity, scarcely differing from a line" (*Conic Sections* 1; *OM* 1:297).

Hobbes found this confused jumble of doctrine the ideal place to launch a counterattack against Wallis. He was particularly emphatic in pressing the point that the infinitely narrow parallelograms that allegedly compose surfaces must each have either some altitude or none. But in either case the doctrine faces incoherence:

> The least Altitude, is Somewhat or Nothing. If Somewhat, then the first character of your Arithmeticall Progression must not be a cypher; and consequently the first eighteen Propositions of this your *Arithmetica Infinitorum* are all naught. If Nothing, then your whole figure is without Altitude, and consequently your Understanding naught. (*SL* 5; *EW* 7:308)

Hobbes found it particularly galling that Wallis should occasionally resort to the expedient of claiming that the altitude of these infinitely small parallelograms need not be considered, and that in such cases the infinitesimal parallelogram is taken for a line.[45] This manner of expression differs very little from Hobbes's doctrine that lines are bodies whose length is considerable but whose breadth need not be considered, and he reminds Wallis that " 'Tis very ugly in one that so bitterly reprehendeth a doctrine in another, to be driven upon the same himself by the force of truth when the thinks not on't" (*SL* 5; *EW* 7:309).

Wallis's departure from Cavalieri's cautious approach to matters of the infinite made him vulnerable to charges that his methods lacked an adequate foundation. Never sparing an opportunity to challenge the competence of his opponent, Hobbes proceeded to level such charges in the first of his *Three Papers Presented to the Royal Society, against Dr. Wallis* and in subsequent publications directed at the Royal Society, including the *Lux Mathematica*. In the *Lux Mathematica* Hobbes

45. Wallis, quoted by Hobbes, declares, "*We will sometimes call those Parallelograms rather by the name of Lines then of Parallelograms, at least, when there is no* consideration *of a determinate Altitude; But where there is a* consideration *of a determinate Altitude (which will happen sometimes) there that little Altitude shall be so far* considered, *as that being infinitely multiplied it may be equal to the* Altitude *of the whole Figure*" (*SL* 5; *EW* 7:309).

summarizes his case against Wallis's results by arguing that they all rest upon two fundamentally mistaken principles:

> The first is one that, so he says, comes from Cavalieri, namely this: ***that any continuous quantity consists of an infinite number of indivisibles***, or of infinitely small parts. Although I, having read Cavalieri's book, remember nothing of this opinion in it, neither in the axioms, nor in the definitions, nor in the propositions. For it is false. A continuous quantity is by its nature always divisible into divisible parts: nor can there be anything infinitely small, unless there were given a division into nothing. (*Lux* 3; *OL* 5:109)

This objection is essentially an analogue of the claim that no collection of breadthless lines can be made to compose a surface, except that it is now phrased as the impossibility of dividing a continuous magnitude into indivisible parts. As such, it highlights the grave conceptual difficulties standing in the way of a mathematically sound theory of infinitesimal magnitudes, as well as the extent to which Wallis departs from Cavalieri's statements on the infinite.

The second flawed principle that Hobbes detects at the basis of Wallis's quadratures is one "so absurd that I can scarcely believe it is advanced by a sane man" (*Lux* 3; *OL* 5:110). The principle, as Hobbes states it, amounts to the assertion that an infinite series of quantities can be understood to have an end, or that the infinite can be taken to be finite. The basis for this charge against Wallis lies in his practice of taking two infinite sums of quantities and concluding that, in the infinite case, a determinate ratio will arise between them. Consider, for example, the thirty-ninth proposition in the *Arithmetica Infinitorum*. We have already examined the proposition briefly in chapter 1 but it is worth another look, particularly because Hobbes himself cites it as a basic example of Wallis's incoherent methods (*Lux* 3; *OL* 5:109). In the proposition Wallis attempts to establish the ratio between a sum of cubic quantities and a sum of quantities equal to the greatest cubic quantity and equal in number to them. Following his usual "inductive method" Wallis begins by examining the first few finite cases of the sequence, and observing that they tend toward the ratio 1:4. He then concludes that

> if an infinite series is taken of quantities in triplicate ratio to a continually increasing arithmetical progression, beginning with 0 (or, equivalently, if a series of cube numbers is taken) this will

> be to the series of numbers equal to the greatest and equal in number as one to four. (*AI* 41; *OM* 1:382–83)

A major difficulty with this reasoning is the supposition that there can be a completed infinite series of quantities "equal in number" to another infinite series. Hobbes complains that Wallis requires that there be a last term to each of the infinite series compared in the ratio, which is in effect the assumption that the infinite can be understood as limited or finite (*Lux* 3; *OL* 5:110; *EW* 7:443). The reasoning is actually doubly problematic because the denominator of the fraction must, in the infinite case, be an infinite sum of infinitely great quantities.

The infinite as understood by Hobbes must always be inexhaustible, essentially incomplete, and beyond our comprehension. In *Leviathan* he declares that "[w]hatsoever we imagine is *Finite*. Therefore there is no Idea, or conception of anything we call *Infinite*. . . . When we say any thing is infinite, we signifie onely, that we are not able to conceive the ends, and bounds of that thing named; having no conception of the thing, but of our own inability" (*L* 1.3, 11; *EW* 3:17). This doctrine has clear precedents in the classical conception of the infinite, and it is linked in Hobbes's scheme of things to a strongly empiricist epistemology in which all ideas must take their origin in sense experience.[46] Because there can be no sensory experience of something infinitely large or small, the most that can be understood by the term "infinite" is the lack of limits. But when Wallis takes an infinite sum of quantities as a given whole, and compares it in a ratio with an infinite sum of quantities (each of which itself is infinite), he in effect treats the infinite as finite, or the limitless as limited.

The issue in dispute here is fundamental to Wallis's whole procedure, and it was debated by numerous others besides Hobbes and Wallis. Indeed, controversy over the nature and status of infinitary methods is a theme that recurs throughout the history of mathematics. In this conflict, Hobbes upholds an essentially classical attitude toward the infinite. He denies that the concept is fully comprehensible and requires that a properly developed mathematical theory make do with a conception of the infinite as inexhaustible and incomplete. This does not mean that Hobbes leaves no room for the infinite in his mathematics. He can grant, for example, that there are infinitely many numbers

46. As Hobbes states in *Leviathan*, "The Originall of them all, is that which we call SENSE; (For there is no conception in a mans mind, which hath not at first, totally, or by parts, been begotten upon the organs of Sense.) The rest are derived from that originall" (*L* 1.1, 3; *EW* 3:3).

or an infinite number of triangles on a given base, and he would allow such claims to be stated and proved within a mathematical theory. Nevertheless, he refuses to allow actually infinite collections to be taken as completed wholes.

It is worth remarking that some of Hobbes's procedures in *De Corpore* lend themselves fairly easily to an interpretation that seems at variance with the rejection of the infinite we find in other of his writings. For instance, some of his attempts to square the circle and his quadratures of deficient figures occasionally speak of quantities being divided infinitely, of figures being made up "of indivisible spaces" and of a "last part" of a division of a magnitude being found.[47] It should be remembered, however, that Hobbes's definitions of such terms as *point* and *line* allow for points to have some magnitude (although the magnitude is not considered), and lines to have some breadth (again, this remaining unconsidered). Thus, Hobbes can say that any quantity can be divided into two quantities of the same kind (i.e., there is no least part of a magnitude), but that beyond a certain threshold the resulting magnitudes can be left out of consideration (i.e., there will be some finite part remaining although its magnitude cannot be considered in a demonstration). This permits Hobbes to reject the actual division of magnitudes to infinity while retaining the language of "least parts" and taking the concept of the infinite to be indefinite or essentially incomplete.

Wallis, of course, wants to make the infinite an object of legitimate mathematical inquiry, and in this he represents the "progressive" mathematicians of the seventeenth century whose work culminated in the development of the calculus. But even Wallis did not wish to break completely with the classical tradition. In the fifth chapter of his *Mathesis Universalis,* during a discussion of the nature of number, he remarks: "Since the addition of units can be extended infinitely, nor can there ever be a greatest number because one or more units can be joined to it, it is therefore impossible that a greatest number be assigned, and still less every species of number. And thus although it is impossible that there should be numbers actually infinite, or an infinite number (that is which has no termini but exceeds all limits), nevertheless an infinity of numbers are possible" (*MU* 5; *OM* 1:28). Hobbes was naturally unimpressed by this apparent concession to the more classical conception of the infinite as incomplete, and he remarked that

47. Examples of this sort of language can be found in the appendix, particularly sections 2 and 3.

"even boys know" that there cannot be a greatest number, although he wondered "how boys can know this when they hear geometers themselves speak of a right line divided into parts infinite in number, and read Wallis's book *Arithmetica Infinitorum*?" (*Examinatio* 1; *OL* 4:52). Hobbes's complaints against Wallis's use of the infinite were summarized in the first of his *Three Papers* presented to the Royal Society, which concluded that the procedures employed in the *Arithmetica Infinitorum* were not only ill founded but useless for their intended purpose.

Wallis attempted to deflect Hobbes's criticisms by arguing that he never asserted the existence of an actual infinite. Instead, he claimed to have reasoned hypothetically, showing that if the completion of an infinite process is supposed, then the required result follows. As he put the matter in his reply to Hobbes's *Three Papers:* "Whether those things *Be* or *Be not;* yea, whether they *Can* or *Cannot be;* the Proposition is not at all concerned, (which affirms nothing either way;) but, whether they can be ***supposed***, or made the ***supposition, in a conditional Proposition.*** As when I say, *If* Mr. Hobs ***were a Mathematician, he would argue otherwise;*** I do not affirm that either ***he his***, or ever ***was*** or ***will be*** such. I only say (upon such supposition) ***If he were***, what he is not; he would not do as he doth" (Wallis 1671, 2241). The stunning ineptitude of this response is worth noting. Wallis seems entirely unconcerned by the possibility that by supposing something inconsistent or impossible, he would deprive his results of an adequate demonstration. Hobbes pointed out exactly the difficulty here, namely that "if the supposition be impossible, then that which follows will either be false or at least undemonstrated" (Hobbes 1671a, 1; *EW* 7:443). Of course, it is the very possibility of an infinite sequence with a last member that is at issue here, and Wallis makes no headway against the objection; his insistence that the completion of an infinite sequence can be harmlessly supposed simply begs the question against those who challenge the coherence of infinitesimal mathematics.

Their opposing views on the infinite crystallized in the two disputants' different evaluations of Torricelli's counterintuitive discovery of the "acute hyperbolic solid" that has an infinite length but a finite volume.[48] This result, first communicated by Torricelli in 1643, involves the solid generated from the hyperbola $xy = a^2$ by first revolving it about one of its asymptotes and cutting it by a plane perpendicular to

48. This theorem and its connection to the Hobbes-Wallis controversy have been studied by Mancosu and Vailati (1991). My treatment of the issue is consequently brief.

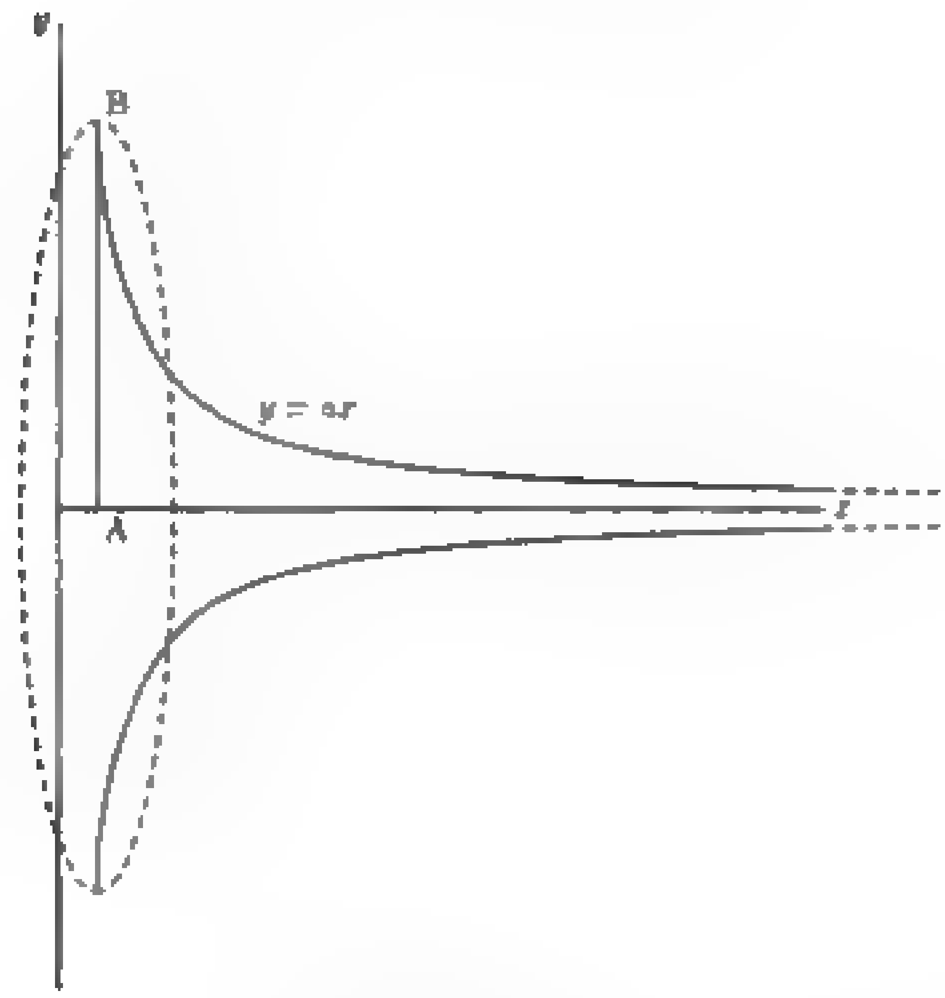

Figure 4.7

the axis of revolution. Thus, in figure 4.7, the hyperbola $y = ax$ is cut by a plane through AB and rotated about the x axis, producing a solid of infinite length but finite volume. Wallis cited the result to show that there is no difficulty in supposing an infinite quantity to be completed, bounded, or to have a last member. He points out that "[a] surface, or solid, may be *supposed* to be so constituted, as to be *Infinitely Long,* but *Finitely Great,* (the Breadth continually Decreasing in greater proportion than the Length Increaseth,) and so as to have *no Center of Gravity.* Such is *Toricellio's Solidum Hyperbolicum acutum;* and others innumerable, discovered by Dr. *Wallis,* Monsieur *Fermat,* and others. But to determine this, requires more of *Geometry and Logick* than Mr. *Hobs* is Master of" (Wallis 1671, 2243).

Hobbes's reaction was to insist that Torricelli's result could not be taken to assert the existence of an actually infinite solid of finite volume, for to understand it in that sense "it is not required that a man should be a geometrician or a logician, but that he should be mad" (Hobbes 1671a, 1; *EW* 7:445). Instead, Hobbes tried to show that Torricelli's theorem could be understood by taking the infinite to be indefinite and incomplete. In the final chapter of his *Principia et Problemata aliquot Geometrica,* which bears the title "On the infinite," Hobbes considers the Torricellian result from the standpoint of his doctrine of the infinite. He insists that "the infinite is the same as the unfinished *[imperfectum],* and is neither complete *[finitum],* nor can it

ever be finished" (*PPAG* 13; *OL* 5:211). In the case of mathematics, Hobbes claims that the infinite should be understood as the indefinite, or that which can be as great or small as desired. He concludes that "nothing is properly signified by the word *infinite*, except it exceeds every assignable number of given measures [*superet mensurarum datarum numerum omnem assignabilem*]" (*PPAG* 13; *OL* 5:213).

With particular reference to the case of the Torricellian solid, Hobbes considers the objection that the acute hyperbolic solid shows that the infinite can be taken as a fixed and determined quantity, although one greater than any assignable one. He replies that

> the distance which Torricelli supposes *infinite* is to be understood as *indefinite*. Nor could it be understood differently by him, who in quite a lot of demonstrations uses Cavalieri's principles of indivisibles, which are such that their aggregate can be equated to whatsoever given magnitude. Therefore, a proposition as absurd as this, that the infinite is equal to the finite, must not be attributed to Torricelli. In fact, as is clear by the natural light, there can be no solid so subtle that does not infinitely exceed every finite solid. (*PPAG* 13; *OL* 5:213)

This reading of the result runs counter to Torricelli's own understanding of it, and it is not without difficulties of its own. Most obviously, taking the solid as only indefinitely long requires that the equality in the theorem be restated somewhat. Rather than saying that it is of indefinite (finite) length and equal in volume to a given cylinder, Hobbes must say that the solid can be made as long as desired and, in consequence, its volume will approach that of the associated cylinder to within any desired degree of accuracy. Hobbes's own comments are sufficiently brief to make it unclear just how he intends the theorem to be restated, but it is clearly difficult for him to take it at face value.[49]

In the end, this phase of the dispute shows that Hobbes was in some

49. On this point Mancosu and Vailati remark that "one could construct an infinite succession of volumes of such solids and claim, by a limit process, that the succession converges to the volume of the cylinder. However, attributing such a view to Hobbes would not only be historically inaccurate, because it would involve forcing our own mathematical notions on him, but would also foist on him conclusions he would hardly tolerate. In fact, although Hobbes would certainly accept the hyperbolic solids whose volumes are elements of this succession, he would have to reject the passage to the limit, since the limit of the succession does not belong to the succession itself, and in order to be equal to the volume of the cylinder the limit must be equal to the volume of an actually infinitely long solid. In sum, Torricelli's result is boldly infinitistic, and Hobbes's attempts at reducing it to a finitistic framework are destined to fail" (1991, 68–69).

important respects more scrupulous and rigorous than his antagonist. Although they almost invariably achieve results that we today recognize as valid, Wallis's methods were shot through with inconsistencies and incoherence, while his pronouncements on the foundations were more obfuscatory than clarificatory. Hobbes rightly pointed out the obscurity of infinitesimal mathematics, and although he did not have a fully developed alternative, his objections were not the ravings of a madman.

CHAPTER FIVE

The "Modern Analytics" and the Nature of Demonstration

> Therefore this analytics is an altogether narrow thing, although it is not completely useless in trigonometry applied to right lines; but because of the great multitude of symbols with which it is burdened today, along with the false opinion that values this method more than it truly merits, it is to be regarded as the plague of geometry.
>
> —Hobbes, *Examinatio*

The disputes outlined in chapter 4 all concern issues in the philosophy of mathematics, and our study of them shows that there was a deep and apparently irreconcilable conflict between Hobbes's and Wallis's conceptions of mathematics and the principles appropriate to it. These were by no means the only points of contention between Hobbes and Wallis. They also held opposing views on the role of algebra and the status of analytic geometry, and my principal purpose in this chapter is to explore these differences. This topic is clearly related to the disputes we examined in the previous chapter, but its complexity and importance warrant separate treatment. A proper understanding of the issues in dispute requires that we first examine Hobbes's views on the nature of language and demonstration, including his distinction between analytic and synthetic methods. After covering these topics, we can proceed to consider his specific charges against the "modern analytics" he so frequently and vehemently condemned. Along the way we will also be concerned with Wallis's critique of this part of the Hobbesian enterprise, as well as the objections of others, most notably Descartes and Ward.

5.1 HOBBES ON SIGNS, LANGUAGE, AND DEMONSTRATION

Hobbes's philosophy of mathematics is closely linked to his treatment of language and its role in reasoning. Human beings derive many ben-

efits from the possession and use of language, but chief among them is the fact that language makes demonstrative knowledge, or true science, possible. As he argues in *Elements of Law:* "By the advantage of *names* it is that we are capable of *science,* which beasts, for want of them are not; nor man, without the use of them" (*EL* 1.5.4; *EW* 4:21). Hobbes uses the term *science* to convey something quite different from the meaning typically associated with the term today. The sense of the term relevant here derives from the Latin *scientia* and is meant to apply to knowledge that is certain, infallible, and universally applicable. In addition, Hobbes follows the tradition by requiring that scientific knowledge be grounded in an understanding of the causes of things and developed in the form of demonstrations. The locus classicus for such requirements is Aristotle's account of scientific knowledge as presented in the *Posterior Analytics,* a text that dominated Scholastic discussions of scientific methodology. Despite his scorn for the "school philosophy," it is evident that at least in this aspect of his philosophy Hobbes follows the Aristotelian tradition.[1] In fact, Hobbes himself declares that "this saying of Aristotle is true, 'to know is to know through causes'" (*PPAG* 1; *OL* 5:156).

Hobbes's conception of science therefore commits him to the principle that mathematics, politics, or any other body of knowledge that can be cast *more geometrico* by the use of reasoning and demonstration can count as a science. Hobbes also uses the term "philosophy" for such properly established doctrine, and he links such knowledge to

1. Aristotle's famous definition of scientific knowledge in *Posterior Analytics* 1.2 reads:

> We think we understand a thing *simpliciter* (and not in the sophistic fashion accidentally) whenever we think we are aware both that the explanation because of which the object is is its explanation, and that it is not possible for this to be otherwise. It is clear, then, that to understand is something of this sort; for both those who do not understand and those who do understand—the former think they are themselves in such a state, and those who do understand actually are. Hence that of which there is understanding simpliciter cannot be otherwise.
>
> Now whether there is also another type of understanding we shall say later; but we say now that we do know through demonstration. By demonstration I mean a scientific deduction; and by scientific I mean one in virtue of which, by having it, we understand something. (71b9–19)

The term *explanation* here translates the Greek *αἰτία,* which can also be rendered as "cause," provided that causes are construed broadly to include nonmechanical causes. On Hobbes's conception of science, with particular reference to its roots in Aristotelian and Scholastic teachings, see Leijenhorst 1998 and Gargani 1971.

the understanding of causes in a famous declaration in the fourth part of *Leviathan:*

> By PHILOSOPHY, is understood the Knowledge acquired by Reasoning from the Manner of the Generation of any thing, to the Properties; or from the Properties, to some possible Way of Generation of the same; to the end to bee able to produce, as far as matter, and humane force permit, such Effects as humane life requireth. So the Geometrician, from the Construction of Figures, findeth out many Properties thereof; and from the Properties, new Ways of their Construction, by Reasoning; to the end to be able to measure Land, and Water; and for infinite other uses. So the Astronomer, from the Rising, Setting, and Moving of the Sun, and Starres, in divers parts of the Heavens, findeth out the Causes of Day and Night, and of the different Seasons of the Year; whereby he keepeth an account of Time: And the like of other Sciences. (*L* 4.46, 367; *EW* 3:664)

The emphasis here on practical applications as the end for which all scientific activity is undertaken suggests that Hobbes does not see scientific knowledge as valuable for its own sake, but this is not an issue that I will be addressing. His conception of science as concerned with the understanding of causes is, however, an aspect of Hobbes's general methodology that I will discuss later.

Hobbes draws an important distinction between science (thus understood) and prudence, holding that although both kinds of knowledge depend upon the understanding of signs, they are distinguished by the fact that science is certain, infallible, and universal where prudence is probable, conjectural, and particular. Prudence, which Hobbes equates with accumulated experience, involves the interpretation of natural signs and is something we share with the beasts. In contrast, science involves the imposition of arbitrary signs and the construction of syllogisms to draw necessary conclusions. Thus, where prudential considerations show that dark clouds are a likely sign of impending rain, scientific reasoning unerringly demonstrates that a straight line is the shortest line connecting two points. Hobbes draws a memorable distinction between science and prudence in *Leviathan* when he declares:

> As, much Experience, is *Prudence;* so, is much Science, *Sapience.* For though wee usually have one name of Wisedome for them

> both; yet the Latines did always distinguish between *Prudentia* and *Sapientia;* ascribing the former to Experience, the latter to Science. But to make their difference appeare more cleerly, let us suppose one man endued with an excellent naturall use, and dexterity in handling his armes; and another to have added to that dexterity, an acquired Science, of where he can offend, or be offended by this adversarie, in every possible posture or guard: The ability of the former, would be to the ability of the later as Prudence to Sapience; both usefull; but the latter infallible. (*L* 1.5, 22; *EW* 3:37)

The certainty and universal applicability of science stem from the fact that it is "knowledge of Consequences, and dependence of one fact upon another" (*L* 1.5, 21; *EW* 3:37), whereas prudence is confined to past experience and depends upon a fallible extrapolation from prior cases. In other words, scientific knowledge unerringly derives its conclusions from the true causes of the phenomenon to be explained, while prudential knowledge amounts to little more than guesswork. Another reason for this difference lies in the fact that the signs upon which prudence depends "are but *conjectural;* and according as they have often or seldom failed, so their *assurance* is more or less; but *never full* and *evident*" (*EL* 1.4.10; *EW* 4:17–18). In contrast, the signs used in demonstrative reasoning carry "full evidence" with them, and this evidence stems from the fact that they are imposed at will rather than arising from chance correlations found in nature.[2] It is in this sense that Hobbes can declare science to be "conditionall Knowledge, or Knowledge of the consequence of words," which arises when "Discourse is put into Speech, and begins with the Definitions of Words, and proceeds by Connexion of the same into generall Affirmations, and of these again into Syllogismes" (*L* 1.7, 30; *EW* 3:52–53).

Because science depends upon the use of signs, it must begin with the creation of a scientific language, a process that Hobbes characterizes as the imposition of names.[3] A name is an arbitrarily chosen sensible thing used to aid the recollection in bringing forth the ideas of things named. Hobbes holds that there can be thought without language, but there can be no science (i.e., philosophy) bereft of signs.

2. The contrast between science and prudence is explored further in Barnouw 1990.

3. The literature on Hobbes's theory of language is voluminous. Zarka 1987, pt. 2, is an extensive study of Hobbes's doctrines of meaning, language, and truth, which can

This is because "whatever a man has put together in his mind by ratiocination without such aid will immediately slip from him," and therefore "for acquiring philosophy, some sensible reminders are necessary, by which past thoughts can be recalled and each one registered as in its own order" (*DCo* 1.2.1; *OL* 1:12).

The arbitrary character of linguistic signs is the basis of their demonstrative capacity, and it is a privilege unique to man that he can freely choose the use and meaning of signs and proceed to reason with them. This privilege means that man "can by words reduce the consequences he findes to generall Rules," but it is "allayed by another; and that is the priviledge of Absurdity; to which no living creature is subject, but man onely" (*L* 1.5, 20; *EW* 3:33). Absurdity arises when the signification assigned to terms is not kept constant, or when terms of contrary signification are combined. The principal purveyors of absurd speech are school divines, whom Hobbes accuses of cynically misleading their students in an effort to increase their own power and hide their ignorance. In *De Homine* he contrasts the epistemic position of humans with that of beasts, concluding that deceptions based in the use of language are a disadvantage peculiar to humans:

> Man, if it should please him (and it will please him as often as it will seem to advance his plans), can teach what he knows to be false in given works; that is, he can lie and make the minds of men hostile to the conditions of society and peace. This cannot happen in the societies of other animals, because they judge what is good and bad for them by their senses and not from the complaints of others, the causes of which complaints they cannot understand unless they see them. Moreover, it sometimes happens to those who listen to philosophers and schoolmen that, by the habit of listening, they rashly accept the words they hear even though no sense can be expressed by them (for such are the words that have been invented by the learned to hide their ignorance), and they use them, believing that they are saying something when they say nothing. Finally, because of the ease of speech, the man who really does not think still speaks, and believes that what he says is true, and he can deceive himself. But a beast cannot deceive itself. (*DH* 2.10.3; *OL* 2:91–92)

be supplemented with Hungerland and Vick 1973, Isermann 1991, Robinet 1979, and Sacksteder 1981b. Older, but still useful, are Hostetler 1945, Krook 1956, and Robbe 1960.

Thus, language does not "make man better, but only gives him greater possibilities," since the benefits of language are compensated by the potential dangers to which it exposes humans (*DH* 2.10.3; *OL* 2:92).

Although animals undeniably make noises to communicate, Hobbes holds that "the signification that is made by the voice *[vox]* of one animal to another of the same species is not speech *[sermo]*, because it is not by their free choice *[arbitrio]* but by the necessity of their nature that their voices signify hope, fear, joy, etc., and are expressed by force of these passions" (*DH* 2.10.1; *OL* 2:88). He concludes that other animals must therefore "lack intellect, for intellect is a certain imagination, but one that arises from the agreed signification of words" (*DH* 2.10.1; *OL* 2:89). The use of speech to signify general truths and the intellectual employment of the imagination thus remain unique parts of the human cognitive apparatus.

The names imposed by human convention can be either proper or general: proper names are "singular to one onely thing," while general names are "*Common* to many things" (*L* 1.4, 13; *EW* 3:21). Once names have been instituted they can function either as marks or signs. A mark is a private name used only to remind the speaker of his previous thoughts, while a sign is a name accepted by others and used to communicate thoughts. Scientific knowledge arises when we employ reason to establish true propositions about the world, a proposition being "*a speech consisting of two names copulated, by which the speaker signifies that he conceives the latter name to be the name of the same thing of which the former is the name; or* (which is the same thing) that the former name is comprehended by the latter" (*DCo* 1.3.2; *OL* 1:27). Thus, if I assert the proposition "London is a city," I indicate that I assign the name 'city' to the same thing that I assign the name 'London', while in asserting "Humans are rational," I show that I assign the name 'rational' to whatever I indicate by the name 'humans'. The truth or falsehood of such propositions then depends only upon the relationship between the names employed in it: "When two Names are joyned together into a Consequence, or Affirmation; as thus, *A man is a living creature;* or thus, *if he be a man, he is a living creature,* If the latter name *Living creature,* signifie all that the former name *Man* signifieth, then the affirmation, or consequence is *true;* otherwise *false.* For *True* and *False* are attributes of Speech, not of things" (*L* 1.4, 14; *EW* 3:23).

This conception of truth as dealing with words rather than things runs against the grain of much seventeenth-century thought. Truth was

often reified by thinkers of the period, as when Lord Robert Brooke's treatise *The Nature of Truth* concludes that truth is "the Understanding in its Essence," and undertakes to prove that "the Soule and Truth be One" (Brooke 1641, 33). Even those who resisted such strongly Platonistic accounts of truth held that it involves more than words. Wallis, whose *Truth Tried* is a rebuttal to Brooke, distinguishes a number of different kinds of truth (moral, logical, and metaphysical) and argues that they are "nothing else but an Agreement or Conformity of a *Type* with its *Prototype, Archetypi & Ectypi;* of a Transcript with its Originall; of an *Idea,* or thing representing with that represented; *Signi & Signati*" (Wallis 1642, 2). In this way of thinking about truth it makes sense to speak of "true ideas" as those that agree with their objects; Hobbes rejects such talk as confused holdovers from the Aristotelian notion that "falsity and truth are not in things . . . but in thought" (*Metaphysics* 6.4; 1027b26–27). Hobbes agrees with the Aristotelian tradition that truth and falsity do not reside in things, but rejects the claim that they are in thought. Instead, Hobbes takes truth to be exclusively in words imposed by human convention, so that "the first of all truths had their origin in the arbitrary choice of those who first imposed names on things, or accepted them as imposed by others. So it is true, for example, that man is an animal, for this reason: because it pleased men to impose these two names on the same thing" (*DCo* 1.3.8; *OL* 1:32).

Hobbes conceives of the reasoning by which true propositions are to be established as a kind of arithmetic involving the addition and subtraction of mental contents: "When a man *Reasoneth*, hee does nothing else but conceive a summe totall, from *Addition* of parcels; or conceive a Remainder, from *Substraction* of one summe from another" (*L* 1.5, 18; *EW* 3:29). Such addition or subtraction is performed most easily on signs or words by manipulating them in accordance with purely formal rules, but it is possible to compute without words, in which case our computations will be performed on phantasms. In *De Corpore* (1.1.3; *OL* 1:3–4) Hobbes illustrates this part of his doctrine by an example: suppose a man sees something in the distance, but so indistinctly as not to discern precisely what it is. At this stage he has only the idea of body, for he knows that it must be some kind of visible body. By approaching, he sees that the body moves itself about "now in one place and now in another," and he adds the idea "animated" to his previous idea of body. Upon closer investigation of the animated body he "sees the figure, hears the voice, and perceives other things

that are signs of a rational mind, and has a third idea, even though there is as yet no name for it, namely that by reason of which we call anything rational." The three ideas of body, animated, and rational are then drawn into a mental sum by seeing that they all pertain to the same thing; thus arises a new idea (namely, the idea of man) compounded out of the three ideas of body, animation, and rationality. Analyzing the definition of the concept "man" into those of "body," "animation," and "rationality" is a similar kind of calculating process, but one that involves the decomposition of the complex idea into its components.

So described, reasoning or mental arithmetic is performed on particular ideas and without the use of words. The more usual cases of reasoning, however, involve words rather than ideas of particular things named by words. When words are taken as the object of reasoning the process remains one of calculation, but it consists in the drawing of consequences from the manipulation of general names. "Reason, in this sense," Hobbes writes, "is nothing but *Reckoning* (that is, Adding and Substracting) of the Consequences of generall names agreed upon, for the *marking* and *signifying* of our thoughts; I say *marking* them, when we reckon by our selves; and *signifying*, when we demonstrate, or approve our reckonings to other men" (*L* 1.5, 18; *EW* 3:30).

We have already seen that Hobbes's requirements for scientific knowledge include the stipulation that such knowledge be acquired by the manipulation of arbitrary signs, and we can now see that he locates the generality of such knowledge in the universality of general names. To put the matter another way, Hobbes holds that the use of general names in reasoning is essential if the consequences drawn are to apply beyond the range of past experience. Hobbes famously insists that "experience concludeth nothing universally" (*EL* 1.4.10; *EW* 4:18), and he stresses the crucial role of names in the formation of generalized knowledge. It is by the imposition of agreed-upon general names that we can "turn the reckoning of the consequences of things imagined in the mind into a reckoning of the consequences of Appellations" (*L* 1.4, 14; *EW* 3:21), and thereby extend such consequences of appellations beyond the scope of experience to become general or universal.

Hobbes illustrates the universality of reasoning from names with an imaginary geometrical case: suppose that someone entirely ignorant of speech contemplates a particular triangle and two right angles placed beside it; suppose further that this person concludes that, in this partic-

ular case, the sum of the interior angles is equal to the sum of the two right angles. Hobbes claims that inability to use words makes it impossible for such a person to generalize this result to cover other cases, while someone who has mastered the appropriate geometric vocabulary will acquire truly universalizable knowledge that the same result holds in all cases:

> But he that hath the use of words, when he observes, that such equality was consequent, not to the length of the sides, nor to any other particular thing in his triangle; but onely to this, that the sides were straight, and the angles three; and that that was all, for which he named it a Triangle; will boldly conclude Universally, that such equality of angles is in all triangles whatsoever; and register his invention in these generall termes, *Every triangle hath its three angles equall to two right angles.* (*L* 1.4, 14; *EW* 3:22)

This imaginary case actually echoes Aristotle's example in the *Posterior Analytics,* although it is put to much different purposes. Aristotle had concluded that "since demonstrations are universal, and it is not possible to perceive these, it is evident that it is not possible to understand through perception either; but it is clear that even if one could perceive of the triangle that it has its angles equal to two right angles, we would seek a demonstration and would not, as some say, understand it; for one necessarily perceives particulars, whereas understanding comes by becoming familiar with the universal" (*Posterior Analytics* 1.31; 87b34–39). Hobbes, in contrast, thinks that the universality of demonstrations can be accounted for simply by the use of names and without recourse to universals. I will examine this issue more fully when I consider Hobbes's nominalism and some of its difficulties.

The drawing of consequences from general names is the concern of logic, and Hobbes models his account of scientific inference on the deductive structure of classical syllogistic logic in Aristotle's *Prior Analytics.* A syllogism, in Hobbes's idiom is "speech that consists of three propositions, from two of which the third follows" (*DCo* 1.4.1; *OL* 1:39). In keeping with his account of reasoning as computation, Hobbes treats syllogistic inferences as a kind of mental addition in which the conclusion is drawn as a sum from the two premises (*DCo* 1.4.6; *OL* 1:42). Hobbes observes the standard division of syllogisms into moods and figures, but his treatment of deductive consequence as computation departs rather significantly from the standard view of the

matter.[4] Because science aims to "establish universal rules concerning the properties of things," the syllogisms appearing in a scientific demonstration must contain only general names, for it is "superfluous to consider any other mood in direct figure, besides that in which all the propositions are both universal and affirmative" (*DCo* 1.4.7; *OL* 1:44).

This should suffice as a brief outline of the Hobbesian theory of language and demonstration. There are a number of tensions internal to this account, as well as some important objections that opponents raised to it, and we can now proceed to a consideration of such difficulties. In the end, I think that Hobbes's theory can avoid collapse into outright incoherence, but the full scope of his theory and its implications will emerge only after we have investigated his critique of algebra and analytic geometry. For the present, however, we can concern ourselves with issues that do not bear directly on the status of mathematical theories.

5.1.1 Conventionalism, Causation, and Demonstration

Hobbes's doctrines certainly seem to result in a conception of science that focuses primarily on such purely linguistic activities as the imposition of names, the analysis of meanings through definitions, and the construction of syllogisms. Indeed, Hobbes often writes as if the principal requirement for the acquisition of scientific knowledge is simply to set purely verbal definitions in order and connect their terms by syllogisms. The universal affirmative propositions appearing in scientific syllogisms would seem to depend for their truth only upon speakers' arbitrary conventions about the meanings of general terms, and this fact suggests that the scientist need tend only to the proper arrangement of terms in his syllogisms. In a summary of his account of science in *Leviathan,* Hobbes comes very close to the implausible claim that science involves nothing more than the correct ordering of names:

> Reason is not as Sense, and Memory, borne with us; nor gotten by Experience onely; as Prudence is; but attayned by Industry; first in apt imposing of Names; and secondly by getting a good and orderly Method in proceeding from the Elements, which are Names, to Assertions made by Connexion of one of them to an-

4. This is not the place for a discussion of the theory of consequence as understood in Hobbes's day. See Ashworth 1974, 120–36, for an overview of postmedieval theories of consequence. For studies of Hobbes's logic see Dal Pra 1962 and De Jong 1986.

> other; and so to Syllogismes, which are the Connections of one Assertion to another, till we come to a knowledge of all the Consequences of names appertaining to the subject in hand; and that is it, men call SCIENCE. (*L* 1.5, 21; *EW* 3:35)

Here, Hobbes takes the "Elements" of demonstrative knowledge to be arbitrarily imposed names, and the whole enterprise appears to involve little more than investigating the consequences of such names. This account does have the virtue of making such knowledge completely certain: if demonstrations are grounded in the arbitrary imposition of names and confined to the analysis of names and their interconnections, then (barring inconsistency or ambiguous names) there is no danger that our reasoning might lead to falsehood. On the basis of such pronouncements, it seems that Hobbes is tempted toward an improbable doctrine that secures the truth and certainty of scientific knowledge at the cost of restricting it to the analysis of language. Furthermore, Hobbes's stress on the arbitrariness with which names are "imposed" on the world seems to commit him to a strongly conventionalist conception of truth in which the truth of a proposition amounts to nothing more than speakers' agreement upon the definitions of the terms it contains.[5]

Although Hobbes's stronger statements regarding the role of names may suggest that he conceived of all properly conducted inquiry as involving little more than a manipulation of names, this cannot be the dominant theme in his account of *scientia*. Such an approach clearly conflicts with his insistence that scientific demonstrations must proceed from causes. Hobbes notoriously held that true science must concern itself with the (mechanical) causes of things, whether these be natural phenomena, geometric objects, or the commonwealth. But the search for causes obviously involves more than simply assigning names to things and analyzing definitions. And indeed, Hobbes himself requires that proper scientific definitions contain the causes of the things defined, even to the point of insisting that "names of things that can be understood to have some cause must have this cause or mode of generation in their definitions, as when we define a circle to be a figure arising from the circumlation of a line in a plane, etc." (*DCo* 1.6.13;

5. Sorell 1988, 45–49, argues that Hobbes is not committed to a thoroughly conventionalist theory of scientific truth, although he employs slightly different considerations than those in play here. For an extended study of the role of conventionalism in Hobbes's treatment of science, particularly in the works before *De Corpore*, see Pacchi 1965. Bernhardt 1993 and Meyer 1992, chap. 2, also take up aspects of these questions.

OL 1:72). Elsewhere Hobbes insists that "where there is place for Demonstration, if the first Principles, that is to say, the Definitions contain not the Generation of the Subject; there can be nothing demonstrated as it ought to be" (*SL* epistle; *EW* 7:184). Definitions of this sort cannot be entirely arbitrary or conventional, because it is possible to have a definition that fails to satisfy such a requirement, either by giving no cause of the thing defined or falsely identifying its cause. Thus, we should not take Hobbes's comments on the role of names in science as evidence of a purely conventionalist theory of science or demonstration.[6]

It is admittedly arbitrary and a matter of speakers' convention what word we use to represent any particular thing. But this degree of arbitrariness or conventionality is consistent with there being better or worse definitions of terms, and Hobbes holds that proper definitions are those that reveal the causes of the things defined. For example, speakers of English may use the words *sunrise* or *dawn* interchangeably, and we can agree with Hobbes that it is a quite arbitrary or conventional matter that we use either word to designate a certain type of event. Apart from possible considerations of lyric meter or rhyme scheme, there is really nothing to choose between them. But such arbitrariness in the choice of *words* does not entail that any *definition* is as good as any other. It would be correct (but not fully enlightening) to define either one as "the appearance of the Sun over the horizon." It would be simply incorrect to define either as "the appearance of the Sun over the horizon, as a result of the Sun's revolution about the earth," and indeed the true definition must be "the appearance of the Sun over the horizon, as a result of the earth's circadian rotation." The last of these three definitions is the best precisely because it reveals the true cause of the thing defined, and that it does so is not the result

6. This point has caused some confusion in the literature. Meyer (1992, 95–105) claims that Hobbes's "conventionalist" theory of language and truth runs counter to his insistence that our definitions and reasoning deal with the true natures of things. Pacchi (1965, chap. 8) sees a conflict between Hobbes's "arbitrarism"—the view that truths are arbitrarily imposed by human convention—and his "intuitionism"—the thesis that self-evident first principles can be directly intuited. I agree with Malcolm (1990, 152–53) that "Hobbes's theory of universal truths was a product of his nominalism; and his nominalism was a good deal less extreme than is popularly supposed. He was a nominalist, not an arbitrarist. Hobbes believed that all blue objects, for example, are really similar: our use of the same word to describe them is not a mere freak of human will or fancy. Indeed, his mechanistic theory of sense-perception ensures this, since the nature of the conception in our brains which we connote with the word 'blue' is *caused* directly by the motion of the object which we see. We experience objects as similar because they really do cause similar motions."

of an arbitrary or conventional decision on our part. Hobbes's insistence on the freedom with which humans impose names must therefore be balanced by his recognition that proper definitions express causes. To put the matter another way, when Hobbes insists in *Leviathan* that science must begin with the "apt imposing of Names" we must remember that those names are most aptly imposed when they designate causes, or at least possible causes. Understanding Hobbes's account of definition in this way makes it clear why he can object that "a man may so precisely determine the signification of a word, as not to be mistaken, yet may his Definition be such, as shall never serve for proof of any Theoreme, nor ever enter into any demonstration (such as are some of the Definitions of *Euclide*) and consequently can be no beginnings of Demonstration, that is to say, no Principles" (*SL* 1; *EW* 7:200).

My exposition of Hobbes's program for the philosophy of mathematics in chapter 3 has already considered his view that proper geometric definitions must express the (mechanical) causes of geometric objects. Still, the requirement that demonstrative knowledge proceed from causes may seem to pose a difficulty in the case of mathematics, since we are not in the habit of thinking of this subject as employing causal principles. In fact, however, the insistence upon the causality of mathematical demonstration is hardly a Hobbesian novelty. One notable antecedent is Aristotle's theory of demonstration. Applied to the case of mathematics, the theory of demonstration in the *Posterior Analytics* requires that mathematical proofs proceed from causes.[7] Of course, Aristotelianism has a wider array of causal principles than those we acknowledge today. Because the Aristotelian philosophy takes individual substances as composites of form and matter, its methodology can distinguish between formal, material, efficient, and final causes. Thus, a causal explanation in the Aristotelian tradition can include reference to a substance's form (the formal cause), its matter (the material cause), the process that produced it (the efficient cause), and the end or purpose for which it was produced (the final cause). Given this broad conception of causation, Aristotle required a scientific demonstration be cast in the form of a syllogism whose premises must not only be true, but also express the cause of the conclusion. Demonstra-

7. On Aristotle's account of scientific demonstration, see Hankinson 1995. Mancosu (1992a; 1996, chap. 1) has drawn attention to the issues raised by the Aristotelian theory of demonstration in the context of sixteenth- and seventeenth-century mathematics (including the Hobbes-Wallis controversy). Aristotle's philosophy of mathematics is examined in Apostle 1952, Cleary 1982, Jones 1983, Lear 1982, and Mueller 1970.

tions that do not satisfy the causal condition are termed demonstrations *τοῦ ὅτι*, while those that satisfy it are demonstrations *τοῦ διότι*; these are standardly called "demonstrations of the fact" and "demonstrations of the reasoned fact," respectively. The point of the distinction is that a demonstration *τοῦ ὅτι* shows merely that something is the case, while the demonstration *τοῦ διότι* shows why it is the case by constructing a syllogism whose premises exhibit the cause of the conclusion.

Scholastic philosophy saw extensive debates on the question of whether mathematical demonstrations are true demonstrations, and Hobbes's mathematical and methodological writings should be read against the background of such controversies.[8] One objection raised against the causality of mathematical demonstration argued that the very nature of mathematical objects precluded the possibility of their being the object of demonstrations *τοῦ διότι*; the Jesuit Benedictus Pereira, for example, held that "mathematical things are abstracted from motion, therefore from all kinds of cause" (Pereira 1576, 70). A similar objection is reported by the Jesuit Martin Smigleckius in his *Logica* of 1634, albeit without his endorsement. He comments that some have objected to the scientific status of mathematics on the grounds that "mathematical objects such as quantities and figures, as they are considered by the mathematician, are not in nature, for indeed there are no lines existing per se, or surfaces abstracted from body, or perfect planes, or bodies perfectly round . . . but it is required for a true and perfect demonstration, that it deals with real beings, not imaginary ones" (Smigleckius 1634, 581). True science, according to this sort of objection, is ultimately a science of natural things and must concern itself with objects in the natural order. To the extent that mathematical objects are regarded as abstracted from the natural world, they cannot be the object of scientific investigation. Another kind of argument against the causality of mathematical demonstration held that the procedures used in Euclidean demonstrations could not express true causes because they introduce factors extrinsic to the object whose properties are being demonstrated; the auxiliary lines or circles constructed in the course of a typical proof seem unrelated to the essence of the original object and are therefore poor candidates for causes of the conclusion.

The standard example of this problem, and one discussed by anyone

8. See Crombie 1977, Jardine 1988, Mancosu 1992a, and Mancosu 1996, chap. 1, for more on this debate.

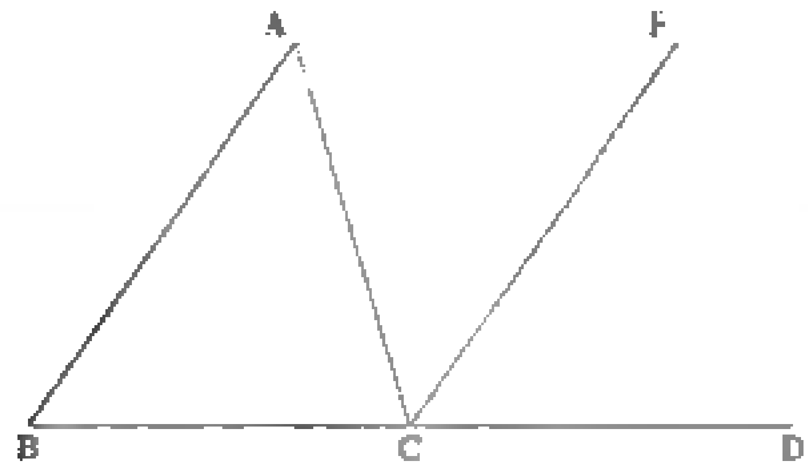

Figure 5.1

who held an opinion on the topic, is Euclid's proof of the theorem that "in any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles" (*Elements* 1, prop. 32). The theorem is, obviously enough, of great foundational importance for Euclidean geometry—almost anyone who discusses the nature or essence of the triangle will mention the crucial fact that its interior angles sum to two right angles. The proof, however, makes use of two auxiliary lines that appear unconnected with the intrinsic nature of the triangle. Let ABC (in figure 5.1) be any triangle, and extend BC to D. Draw CE parallel to BA, and there results an equality of angles $\angle ABC$ and $\angle ECD$, as well as the angles $\angle BAC$ and $\angle ACE$. Thus $\angle ABC + \angle BCA + \angle CAB = \angle ECD + \angle BCA + \angle ACE$, and the result is obtained. Pereira and others who disputed the scientific status of mathematics, including Pierre Gassendi in his *Exercitationes Paradoxicae adversus Aristoteleos*, argued that such a proof fails to appeal to proper causes, since the essential parts of the construction are literally external to the triangle. They and others took this as decisive evidence against the claim that Euclidean geometry could satisfy the Aristotelian criteria for scientific demonstration.[9]

Defenders of the scientific status of mathematics (a group that included Wallis, Barrow, and Clavius) argued that, despite appearances, the proofs in Euclid can be interpreted in a manner consistent with Aristotle's theory of demonstration. Details of these defenses are sufficiently remote from my present concerns that they can be left for another day, but it is important to see that this kind of issue is part of the background to Hobbes's project for the philosophy of mathematics.

9. For an account of Gassendi's use of this argument from Pereira, see Mancosu 1996, 19–24. The principal argument for the thesis that geometry fails to meet the Aristotelian standard for science relies on Pereira and can be found in Gassendi [1658] 1964, 3:208–9.

This particular dispute was a matter of great importance in the philosophy of mathematics of the mid–seventeenth century, as is evident from the fact that Barrow devotes the sixth of his *Lectiones Mathematicae* to the topic of "the causality of mathematical demonstration," while Wallis spends the second and third chapters of his *Mathesis Universalis* arguing that the mathematical sciences count as sciences in the strictest sense. One point to stress here is that the causality of mathematical demonstrations was generally regarded as formal causality, in which the form or essence of a geometric object (as expressed in its definition) is causally responsible for and explanatory of the properties demonstrated of it.[10] In this respect, Hobbes's program for mathematics departs from the classical pattern. Hobbes regarded talk of formal causality as part of the empty verbiage of the schools, and his materialistic program for mathematics requires that the definitions of mathematical objects exhibit the kinds of motions by which such objects are produced. Thus, although Hobbes accepts the traditional dictum that "all knowledge is knowledge of causes," he restricts the concept of causality to that of efficient causality, and even this is understood mechanistically, so that it is only by the motion and impact of material bodies that anything can be caused. It is therefore evident that Hobbes's concerns with the causality of mathematical demonstration are anchored in Scholastic and early modern debates over the status of the mathematical sciences, but his solution to the problem departs significantly from the tradition.

His methodological principles also led Hobbes to downplay the distinction between demonstrations *τοῦ διότι* and *τοῦ ὅτι*. In his *Mathesis Universalis* Wallis had invoked the standard distinction between *τοῦ διότι* and *τοῦ ὅτι* in the course of defending the truly scientific nature of mathematics, arguing that many demonstrations were of the most perfect kind, which he calls "ostensive τοῦ διότι."[11] Hobbes objected

10. Barrow, in discussing the causality of mathematical proofs, sums up his opinion on this issue: "Such in truth, and no other, is the causality and mutual dependence of the terms of a mathematical demonstration. That is, a most close and intimate connection of them with one another, which can always be called formal causality, in that from one property first assumed, other attributes *(passiones)* result as from a form. Nor do I think that there is any other causality in the nature of things, in which a necessary consequence may be founded" (*LM* 6, 93).

11. The *Mathesis* states, "But a third sort of demonstration, which is the most perfect of all, is ostensive *τὸ διότι*, which demonstrates both that something is and why it is. A demonstration is of this sort if someone demonstrates that all the radii of the same circle are equal from the fact that the circle is defined (or at least can be defined) as a plane figure contained within one curve that is everywhere equidistant from the middle